24. *Singularities and Cauchy Problem for Fuchsian Hyperbolic Equations*

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In this paper, we shall discuss distribution solutions of the Cauchy problem for Fuchsian hyperbolic equations (in Tahara [2], Bove-Lewis-Parenti [1]), and investigate the propagation of singularities of them by using the notion of wave front sets. The result here is a generalization of results in [1].

1. Fuchsian hyperbolic equations. Let us consider the Cauchy problem:

$$\text{(E)}\qquad \begin{cases} t^k\partial_t^m u+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} t^{p(j,\alpha)}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha u=f(t,x),\\ \partial_t^i u|_{t=0}=g_i(x), \qquad i=0,1,\cdots,m-k-1,\end{cases}$$

where $(t,x)=(t,x_1,\cdots,x_n)\in[0,T]\times\boldsymbol{R}^n\ (T>0)$, $m\in\boldsymbol{N}(=\{1,2,\cdots\})$, $k\in\boldsymbol{Z}_+$ $(=\{0,1,2,\cdots\})$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\boldsymbol{Z}_+^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $p(j,\alpha)\in\boldsymbol{Z}_+(j+|\alpha|\leqq m$ and $j<m)$, $a_{j,\alpha}(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ and $j<m)$, $\partial_t=\partial/\partial t$, and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. In addition, we impose the following conditions (A-1)~(A-3) on (E).

(A-1) $0\leqq k\leqq m$.

(A-2) $p(j,\alpha)\in\boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy

$$\begin{cases} p(j,\alpha)=k+\nu|\alpha|, & \text{when } j+|\alpha|=m \text{ and } j<m,\\ p(j,\alpha)\geqq k-m+j+(\nu+1)|\alpha|, & \text{when } j+|\alpha|<m\end{cases}$$

for some $\nu\in\boldsymbol{Z}_+$.

(A-3) All the roots $\lambda_i(t,x,\xi)$ $(i=1,\cdots,m)$ of

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=0$$

are *real*, *simple* and *bounded* on $\{(t,x,\xi)\in[0,T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n;\ |\xi|=1\}$.

Then, the equation is one of the most fundamental examples of Fuchsian hyperbolic equations. Therefore, by applying the result in Tahara [2] we can obtain the C^∞ well posedness of (E), that is, the existence, uniqueness and finiteness of propagation speed of solutions in $C^\infty([0,T]\times\boldsymbol{R}^n)$. To prove these results, Tahara [2] used the energy inequality method.

Recently, Bove-Lewis-Parenti [1] has succeeded to construct a right and a left parametrix for the case $\nu=0$, and obtained the existence,

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uniqueness and propagation results of singularities of distribution solutions.

In this note, we want to report that the method developed in [1] can be applied to the general case $\nu \geqq 0$.

2. Existence and uniqueness. Let $\mathcal{D}'(\boldsymbol{R}^n)$ be the locally convex space of all distributions on $\boldsymbol{R}^n$ with strong topology, and let $C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ be the space of all infinitely differentiable functions on $[0, T]$ with values in $\mathcal{D}'(\boldsymbol{R}^n)$.

Let $\mathcal{C}(\rho, x)$ be the characteristic polynomial of (E), that is, $\mathcal{C}(\rho, x)$ is defined by

$$\mathcal{C}(\rho, x)=\rho(\rho-1)\cdots(\rho-m+1)+a_{m-1}(x)\rho(\rho-1)\cdots(\rho-m+2)$$
$$+\cdots+a_{m-k}(x)\rho(\rho-1)\cdots(\rho-m+k+1),$$

where

$$a_j(x)=\begin{cases} a_{j,(0,\dots,0)}(0, x), & \text{when } p(j, (0, \cdots, 0))=k-m+j, \\ 0, & \text{when } p(j, (0, \cdots, 0))>k-m+j. \end{cases}$$

In order to solve (E) at a formal power series level, we impose the following condition.

(A-4) $\mathcal{C}(\rho, x)\neq 0$ for any $x \in \boldsymbol{R}^n$ and $\rho \in \{\lambda \in \boldsymbol{Z};\ \lambda \geqq m-k\}$.

(In [1], this condition is called the Fuchs condition.)

Then, by constructing a right and a left parametrix for (E) like as in Bove-Lewis-Parenti [1] we can solve (E) in $C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ modulo $C^\infty([0, T]\times \boldsymbol{R}^n)$. Hence, by combining this with the C^∞ well posedness in Tahara [2] we can obtain the following result.

Theorem 1. *Assume that* (A-1)~(A-4) *hold. Then, for any* $f(t, x) \in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ *and any* $g_i(x) \in \mathcal{D}'(\boldsymbol{R}^n)$ $(i=0, 1, \cdots, m-k-1)$ *there exists a unique solution* $u(t, x) \in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ *of* (E). *Moreover, the domain* $D(t_0, x^0)$ *defined by*

$$D(t_0, x^0)=\{(t, x) \in [0, T]\times \boldsymbol{R}^n;\ |x-x^0|<\lambda_{\max}T^\nu(t_0-t)\}$$

(*where* $\lambda_{\max}=\sup\{|\lambda_i(t, x, \xi)|;\ i=1, \cdots, m,\ (t, x) \in [0, T]\times \boldsymbol{R}^n$ *and* $|\xi|=1\}$) *is a dependence domain of* $(t_0, x^0) \in (0, T]\times \boldsymbol{R}^n$. *In other words, if* $f(t, x)=0$ *on* $D(t_0, x^0)$ *and* $g_i(x)=0$ *on* $D(t_0, x^0)\cap\{t=0\}$ $(i=0, 1, \cdots, m-k-1)$ *hold, then* $u(t, x)$ *also satisfies* $u(t, x)=0$ *on* $D(t_0, x^0)$.

3. Singularities of solutions. We say that $f(t, x) \in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ is a regular distribution, if $f(t, x)$ satisfies

$$WF(f|_{t>0})\cap\{(t, x, \tau, \xi)|\ t>0,\ \xi=0\}=\phi.$$

For a regular distribution $f(t, x)$, we define the boundary wave front set $\partial WF(f)$ $(\subset T^*\boldsymbol{R}^n\backslash 0)$ over $\{t=0\}$ by the following; we say that a point $(x, \xi) \in T^*\boldsymbol{R}^n\backslash 0$ does not belong to $\partial WF(f)$, if and only if there exists a classical pseudo-differential operator $B(x, D_x)$, elliptic near (x, ξ), such that $(Bf)(t, x) \in C^\infty([0, \varepsilon]\times \boldsymbol{R}^n)$ for some $\varepsilon>0$.

Let $\nu \in \boldsymbol{Z}_+$ be as in (A-2), let $\lambda_i(t, x, \xi)$ $(i=1, \cdots, m)$ be as in (A-3), and let $(x^{(i)}(t, s, y, \eta), \xi^{(i)}(t, s, y, \eta))$ be the solution of the ordinary differential equation (in t)

$$\frac{dx^{(i)}}{dt} = -t^\nu \nabla_\xi \lambda_i(t, x^{(i)}, \xi^{(i)}), \qquad \frac{d\xi^{(i)}}{dt} = t^\nu \nabla_x \lambda_i(t, x^{(i)}, \xi^{(i)}),$$
$$x^{(i)}|_{t=s} = y, \qquad \xi^{(i)}|_{t=s} = \eta.$$

Then, by investigating directly the right or the left parametrix for (E) we can obtain the following result.

Theorem 2. *Assume that* (A-1)$\sim$(A-4) *hold. Let* $u(t, x) \in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ *be the unique solution of* (E) *in Theorem* 1, *and assume that* $f(t, x)$ *is a regular distribution. Then,* $u(t, x)$ *is also a regular distribution and the following inclusions hold.*

(1) $$\partial WF(u) \subset \partial WF(f) \cup \bigcup_{j=0}^{m-k-1} WF(g_j).$$

(2) $$\begin{aligned} WF(u|_{t>0}) \subset & \{(t, x, \tau, \xi) \mid t>0,\ (t, x, \tau, \xi) \in WF(f)\} \cup \\ & \bigcup_{i=1}^{m} \Big\{(t, x, t^\nu \lambda_i(t, x, \xi), \xi) \mid t>0,\ \exists s,\ \frac{s}{t} \in (0, 1),\ \exists (y, \eta) \in T^*\boldsymbol{R}^n \backslash 0, \\ & \qquad x = x^{(i)}(t, s, y, \eta),\ \xi = \xi^{(i)}(t, s, y, \eta), \\ & \qquad (s, y, s^\nu \lambda_i(s, y, \eta), \eta) \in WF(f)\Big\} \cup \\ & \bigcup_{i=1}^{m} \Big\{(t, x, t^\nu \lambda_i(t, x, \xi), \xi) \mid t>0,\ \exists (y, \eta) \in T^*\boldsymbol{R}^n \backslash 0, \\ & \qquad x = x^{(i)}(t, 0, y, \eta),\ \xi = \xi^{(i)}(t, 0, y, \eta), \\ & \qquad (y, \eta) \in \partial WF(f) \cup \bigcup_{j=0}^{m-k-1} WF(g_j)\Big\}. \end{aligned}$$

Details and proofs will be published elsewhere.

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