23. On the Global Existence of Real Analytic Solutions and Hyperfunction Solutions of Linear Differential Equations

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§0. The purpose of this note is to show that the microlocal analysis enables us to derive global existence theorems for hyperfunction solutions and real analytic solutions from the semi-global existence theorem for real analytic solutions due to Kiro [2].

The author would like to express his heartiest thanks to Prof. Y. Kannai for having kindly called his attention to the paper [2] before its publication.

Although several parts of the results described below can be proved for a wider class of equations, here we discuss only single equations with real simple characteristics, for the sake of simplicity. Details of this note shall appear somewhere else.

§1. In Corollary of Theorem 5.4, Kiro [2] proved that a linear differential operator P acting on the space $\mathcal{A}(K)$ of real analytic functions on K is of closed range with finite codimension for a compact set $K=\{x \in \boldsymbol{R}^n ; \Psi(x) \leqq 0\}$ (Ψ ; C^∞ and $d\Psi|_{\partial K} \neq 0$), if

(1.1) the principal symbol $p_m(x, \xi)$ of P is real for real (x, ξ) and $\operatorname{grad}_\xi p_m$ never vanishes if $p_m=0$ and $\xi \neq 0$,

and

(1.2) for any characteristic boundary point x^0, the bicharacteristic curve $\{x(\tau)\}$ emanating from $(x^0, \operatorname{grad}\Psi(x^0))$ satisfies the following:

$$\left.\frac{d^2\Psi(x(\tau))}{d\tau^2}\right|_{\tau=0} > 0.$$

As our subsequent reasoning needs the surjectivity of $P: \mathcal{A}(K) \to \mathcal{A}(K)$, we first prepare the following Proposition 1.1. Here, and in what follows, we denote by $\mathring{T}^*U$ the cotangent bundle of U deleted its zero-section, i.e., $\mathring{T}^*U = T^*U - T^*_U U$.

Proposition 1.1. *Suppose that the pair (P, K) satisfies the following condition* (1.3) *in addition to* (1.1) *and* (1.2):

(1.3) *For any (x, ξ) in $\mathring{T}^*\boldsymbol{R}^n$ satisfying $p_m(x, \xi)=0$ with x in K, we can find a bicharacteristic curve b of p_m which passes through (x, ξ) and (y, η) $(\eta \neq 0)$ with y outside K.*

Then $P: \mathcal{A}(K) \to \mathcal{A}(K)$ is surjective.

Since the dual space of $\mathcal{A}(K)$ is the space $\mathcal{B}_K$ of hyperfunctions supported by K, the closed range theorem combined with the general result on the propagation of singularities ([1], [3]) entails the surjectivity of P.

§ 2. Thanks to the flabbiness of the sheaf of hyperfunctions, we can employ the microlocal analysis to derive the following global existence theorem (Theorem 2.1 below) from the semi-global existence theorem.

Theorem 2.1. *Let Ω be a relatively compact open subset of $\mathbf{R}^n$ which has the form $\{x \in \mathbf{R}^n ; \Psi(x)<0\}$ for a real-valued real analytic function $\Psi(x)$ defined on a neighborhood of the closure K of Ω satisfying* $\mathrm{grad}_x \Psi(x) \neq 0$ *on $\partial\Omega$. If the pair (P, K) satisfies conditions* (1.1), (1.2), (1.3) *and the condition* (2.1) *below, then $P : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective.*

(2.1) *There exists a finite family of open sets $\{N_j\}_{j=1}^{p}$ which satisfies the following condition: For each point x in $\partial\Omega$ there exists j such that N_j is a neighborhood of x and that, for any bicharacteristic curve $b_{(x,\xi)}$ of p_m passing through (x, ξ) satisfying* $p_m(x, \xi)=0$ $(\xi \neq 0)$ *and* $\langle \mathrm{grad}_x \Psi(x), \mathrm{grad}_\xi p_m(x, \xi)\rangle = 0$, *$b_{(x,\xi)} \cap \Omega \cap N_j = \phi$ holds.*

In fact, by the aid of the flabbiness of the sheaf of hyperfunctions and that of microfunctions, we can use the condition (2.1) to deduce the surjectivity of $P : \mathcal{A}(\Omega)/\mathcal{A}(K) \to \mathcal{A}(\Omega)/\mathcal{A}(K)$ from the general structure theorem for microdifferential equations. ([3], Chap. III.) Then Proposition 1.1 guarantees the surjectivity of $P : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$.

§ 3. To prove the global existence of hyperfunction solutions, let us first discuss $(\mathcal{B}/\mathcal{A})$-solutions. Let P be a linear differential operator defined on an open subset U of $\mathbf{R}^n$, and suppose that it satisfies the condition (1.1). Let $\varphi(x)$ be a real-valued real analytic function defined on U, and define the set $U(t)$ by $\{x \in \mathbf{R}^n ; \varphi(x)<t\}$ $(0<t<\infty)$. Suppose that $U(t)$ satisfies the following conditions:

(3.1) $\bigcap_{0<t} U(t)$ consists of one point in U,

(3.2) for any $t, \varepsilon>0$ and $\varepsilon' \geq 0$ $(\varepsilon, \varepsilon' \ll 1)$, there exist two conic closed subsets $W_\pm(t, \varepsilon, \varepsilon')$ of $\mathring{T}^*U$ which satisfy the following:

(3.2.a) $W_+(t, \varepsilon, \varepsilon') \cup W_-(t, \varepsilon, \varepsilon') = \{(x, \xi) \in \mathring{T}^*U ; x \in \overline{U(t+\varepsilon')} \setminus U(t-\varepsilon), p_m(x, \xi)=0\}$

(3.2.b) for any point (x, ξ) in $W_+(t, \varepsilon, \varepsilon')$ (resp., $W_-(t, \varepsilon, \varepsilon')$) the positive part $b^+_{(x,\xi)}$ (resp., the negative part $b^-_{(x,\xi)}$) of the bicharacteristic curve emanating from (x, ξ) does not intersect $U(t-\varepsilon)$.

Then we have the following

Lemma 3.1. *Let P and φ be as above. Then, for any $t>0$, we find*

$$P : (\mathcal{B}/\mathcal{A})(U(t)) \longrightarrow (\mathcal{B}/\mathcal{A})(U(t))$$

is surjective.

To prove this lemma, it suffices to show

(3.3) $$\sup\{t ; H^1(U(s), (\mathcal{B}/\mathcal{A})^P)=0 \text{ for } 0<s<t\}=\infty.$$

Here, and in what follows, $(\mathcal{B}/\mathcal{A})^P$ denotes the $(\mathcal{B}/\mathcal{A})$-solution sheaf, i.e., $\mathrm{Ker}(P : \mathcal{B}/\mathcal{A} \to \mathcal{B}/\mathcal{A})$. By using the local elementary solution of P, we can verify $H^1(U(s), (\mathcal{B}/\mathcal{A})^P)=0$ for $s \ll 1$. Hence we can use the reduction to the absurdity by assuming the supremum in (3.3) is a finite number c.

Since $H^1(U(c-\varepsilon), (\mathcal{B}/\mathcal{A})^P)$ vanishes for $\varepsilon>0$, we use the long exact sequence for relative cohomology groups to deduce the vanishing of $H^1(U(c+\varepsilon'), (\mathcal{B}/\mathcal{A})^P)$ $(\varepsilon'\geqq 0)$ from that of $H^1_{U(c+\varepsilon')\setminus U(c-\varepsilon)}(U(c+\varepsilon'), (\mathcal{B}/\mathcal{A})^P)$. Since the sheaf $\mathcal{B}/\mathcal{A}$ is flabby, we can easily calculate the relative cohomology group $H^1_{U(c+\varepsilon')\setminus U(c-\varepsilon)}(U(c+\varepsilon'), (\mathcal{B}/\mathcal{A})^P)$. Using condition (3.2), we then deduce its vanishing for any ε' $(0\leqq\varepsilon'\ll 1)$ again from the general structure theorem for microdifferential equations in [3], Chap. III. Since the vanishing of $H^1(U(c+\varepsilon'), (\mathcal{B}/\mathcal{A})^P)$ $(0\leqq\varepsilon'\ll 1)$ contradicts the definition of c, we conclude that c cannot be finite. This implies the surjectivity of $P:(\mathcal{B}/\mathcal{A})(U(t))\to(\mathcal{B}/\mathcal{A})(U(t))$ for every $t>0$.

Combining Lemma 3.1 and Proposition 1.1, we finally obtain the following Theorem 3.2 with the aid of the flabbiness of the sheaf of hyperfunctions.

Theorem 3.2. *Let P and $U(t)$ be the same as in Lemma 3.1. Set $\Psi(x)=\varphi(x)-1$ and suppose that $\{x\in \boldsymbol{R}^n; \Psi(x)\leqq 0\}$ is a compact subset with smooth boundary which satisfies the conditions (1.2) and (1.3). Then, for any open set $\omega\Subset U(1)$, we find*

$$P:\mathcal{B}(\omega)\longrightarrow\mathcal{B}(\omega)$$

is surjective.

References

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