17. On Weak, Strong and Classical Solutions of the Hopf Equation

An Example of F.D.E. of Second Order

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§1. **Introduction and results.** Let (M, g) be a compact Riemannian manifold of $\dim M=d$ with or without boundary ∂M. We denote by $\mathring{X}_\sigma(M)$ the space of solenoidal vector fields on M which vanish near the boundary. $\boldsymbol{H}$ stands for the completion of the above space with respect to L^2-norm, denoted by $|\cdot|$. V^s stands for the completion of $\mathring{X}_\sigma(M)$ in the Sobolev space of order $s \in \boldsymbol{Z}$, whose norm is denoted by $\|\cdot\|_s$. For 1-forms, we introduce $\mathring{\Lambda}^1_\sigma(M)$ analogously. The completions of it with corresponding norms are denoted by $\tilde{\boldsymbol{H}}$ and $\tilde{V}^s$, respectively. The space of symmetric tensor fields with 2 contravariant (or covariant) indices is denoted by $ST_2(M)$ (or $ST^2(M)$.)

Our aim of this paper is to 'solve' the following Functional Derivative Equation (F.D.E.):

(I) Find a functional $W(t, \eta)$, for $t \in (0, \infty)$, $\eta \in \mathring{\Lambda}^1_\sigma(M)$ satisfying

$$\text{(I.1)}\quad \frac{\partial}{\partial t} W(t,\eta)=\int_M \Big[-i\Big\{\frac{\partial}{\partial x^j}\eta_i(x)-\Gamma^l_{ij}(x)\eta_l(x)\Big\}\frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(x)} +\nu(\Delta\eta)_i(x)\frac{\delta W(t,\eta)}{\delta\eta_i(x)}+i\eta_j(x)f^j(x,t)W(t,\eta)\Big]d_g x,$$

$$\text{(I.2)}\quad \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\Big\{\sqrt{g(x)}\frac{\delta W(t,\eta)}{\delta\eta_i(x)}\Big\}=0,$$

$$\text{(I.3)}\quad W(t,0)=1$$

and

$$\text{(I.4)}\quad W(0,\eta)=W_0(\eta).$$

Here $\eta(x)=\eta_j(x)dx^j \in \mathring{\Lambda}^1_\sigma(M)$, and $f(x,t)=f^j(x,t)(\partial/\partial x^j) \in \mathring{X}_\sigma(M)$ for a.e. t, $W_0(\eta)$ is a given positive definite functional on $\mathring{\Lambda}^1_\sigma(M)$ satisfying

$$\text{(I.5)}\quad W_0(0)=1 \quad\text{and}\quad \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^j}\Big\{\sqrt{g(x)}\frac{\delta W_0(\eta)}{\delta\eta_j(x)}\Big\}=0.$$

Hereafter, we use Einstein's convention for contracting indices and also the terminology and symbols from Riemannian geometry and functional analysis. The definition of functional derivatives

$$\frac{\delta W(t,\eta)}{\delta\eta_i(x)} \quad\text{and}\quad \frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(y)}$$

is given, for example, in E. Hopf [3].

A weak solution of Problem (I) will be afforded by considering the following problem.

(II) Find a family of Borel measures $\{\mu_t\}_{0<t<\infty}$ on $\boldsymbol{H}$ satisfying

$$-\int_0^\infty \int_{\boldsymbol{H}} \frac{\partial \Phi(t,u)}{\partial t} d\mu_t(u)dt - \int_{\boldsymbol{H}} \Phi(0,u)d\mu_0(u)$$
$$= \int_0^\infty \int_{\boldsymbol{H}} \int_M \left[\left\{ u^j(x) \frac{\partial}{\partial x^j} u^i(x) + \Gamma^i_{jk}(x) u^j(x) u^k(x) \right\} \frac{\delta \Phi(t,u)}{\delta u^i(x)} \right.$$
$$\left. + \nu \nabla_k u^i(x) \cdot \nabla^k \frac{\delta \Phi(t,u)}{\delta u^i(x)} - f^j(x,t) \frac{\delta \Phi(t,u)}{\delta u^j(x)} \right] d_g x d\mu_t(u) dt$$

for any test functional $\Phi(t,u)$ with compact support in t. The given data are a measure μ_0 and a right member $f(t)$.

Our results are

Theorem A. *For any initial data μ_0, a Borel measure on $\boldsymbol{H}$ satisfying*

$$\int_{\boldsymbol{H}} (1+|u|^2) d\mu_0(u) < \infty$$

and any right term $f(\cdot) \in L^2(0,\infty; V^{-1})$, there exists a solution $\{\mu_t\}_{0<t<\infty}$ of (II).

Theorem B. *Let $W_0(\cdot)$ be a positive definite functional on $\tilde{\boldsymbol{H}}$ and satisfy*

$$\mathrm{trace}_{\tilde{\boldsymbol{H}} \to \boldsymbol{H}} [-W_{0\eta\eta}(0)] < \infty.$$

For any right term $f(\cdot) \in L^2(0,\infty; V^{-1})$, there exists a strong solution of Problem (I).

Theorem C. *Let $\partial M = \phi$ and $l = [d/2]+1$. Let $W_0(\cdot)$ be a positive definite functional on $\tilde{\boldsymbol{H}}$, be of $\tilde{V}^{-l}$-exponential type and satisfy*

$$\mathrm{trace}_{\tilde{\boldsymbol{H}} \to \boldsymbol{H}} [-W_{0\eta\eta}(0)] < \infty \quad \textit{and} \quad \mathrm{trace}_{\tilde{V}^l \to V^l} [-W_{0\eta\eta}(0)] < \infty.$$

For any $f(\cdot)$ given in $L^1_{\mathrm{loc}}(0,\infty; V^l)$, there exists a unique classical solution $W(t,\eta)$ of Problem (I) *on* $[0, T^*)$ *where T^* is defined from W_0 and f, independent of ν.*

Remarks. (1) Technically, we extend the arguments in Foiaş [1, 2] to the case where $T=\infty$ and M is an arbitrary compact Riemannian manifold with or without boundary. Especially, there is no restriction on the dimension d of M. In Theorems A and B, actually M is rather arbitrary, but in Theorem C, we must restrict our attention, to the case where $\partial M = \phi$. (2) We give the strict meaning to the 'trace' of the second order functional derivatives in Problem (I), that is,

$$\frac{\delta^2 W(t,\eta)}{\delta \eta_i(x) \delta \eta_j(x)} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

is defined as a distributional element in $ST_2(M)$, in fairly general situations. This gives the mathematical meaning to the functional derivatives of order 2 appeared in (I.1).

Detailed proofs will be given somewhere else.

§ 2. Definitions and the ideas of the proofs.

Definition. A functional defined on $[0,T) \times \tilde{\boldsymbol{H}}$, $(T \leqq \infty)$ will be called a classical solution of Problem (I) on $(0,T)$ if there exists a set $\tilde{D}$, dense in $\tilde{V}^s$, for some s, containing $\mathring{\Lambda}^1_\sigma(M)$ such that: (1) For each $\eta \in \tilde{D}$, $W(t,\eta)$

is absolutely continuous on $[0, T)$. (2) For each i, j,

$$\frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(x)}$$

exists a.e. t on $[0, T)$ as an element of $L^1_{\mathrm{loc}}(M)$ for each $\eta \in \tilde{D}$. Moreover,

$$\frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(x)}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$$

belongs to $ST_2(M)$. (3) $W(t, \eta)$ satisfies (I.1)–(I.4) a.e. in t as functions for each $\eta \in \tilde{D}$.

Definition. A functional defined on $[0, T)\times\tilde{\boldsymbol{H}}$, $(T\leqq\infty)$ will be called a strong solution of Problem (I) on $(0, T)$ if there exists a set $\tilde{D}$, dense in $\tilde{\boldsymbol{V}}^s$, for some s, containing $\mathring{\Lambda}^1_\sigma(M)$ such that: (1) For each $\eta \in \tilde{D}$, $W(t, \eta)$ belongs to $L^1_{\mathrm{loc}}[0, T)$ and is right continuous in t at $t=0$.

$$(2) \qquad \frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(x)}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$$

exists a.e. t on $(0, T)$ as a distributional element in $ST_2(M)$ for each $\eta \in \tilde{D}$. (3) $W(t, \eta)$ satisfies (I.1)–(I.4) as distributions for each $\eta \in \tilde{D}$.

Definition. A positive definite functional W on $\tilde{\boldsymbol{H}}$ will be called of $\tilde{\boldsymbol{V}}^{-l}$ exponential type for any $\eta \in \tilde{\boldsymbol{H}}$ when the function $s\to W(s\eta)$ defined on $\boldsymbol{R}$ can be extended analytically to an entire function $W(\zeta;\eta)$ on the complex plane $\boldsymbol{C}$ satisfying

$$|W(\zeta;\eta)|\leqq c_1\cdot e^{c_2|\mathrm{Im}\zeta|\|\eta\|_{-l}} \qquad \text{for all } \zeta\in\boldsymbol{C},\ \eta\in\tilde{\boldsymbol{H}},$$

where c_1 and c_2 are some constants depending on W.

Now, we introduce the notion of test functionals.

Definition. A real functional $\Phi(\cdot, \cdot)$ defined on $[0, \infty)\times\boldsymbol{V}$ is called a test functional if it satisfies the followings: (1) $\Phi(\cdot, \cdot)$ is continuous on $[0, \infty)\times\boldsymbol{V}$ and verifies $|\Phi_t(t, u)|\leqq c_3+c_4|u|$. (2) $\Phi(\cdot, \cdot)$ is Fréchet $\boldsymbol{H}$-differentiable in the direction $\boldsymbol{V}$. (3) Moreover, $\Phi_u(\cdot, \cdot)$ is continuous from $[0, \infty)\times\boldsymbol{V}$ to $\tilde{\boldsymbol{V}}^s$ and is bounded. That is, there exists a constant c_5 depending on Φ such that $\|\Phi_u(t, u)\|_s\leqq c_5$ for all $(t, u)\in[0, \infty)\times\boldsymbol{V}$.

We call that a test functional $\Phi(\cdot, \cdot)$ has a compact support in t if there exists a constant T_0 depending on Φ such that $\Phi(t, \cdot)=0$ for $t\geqq T_0$.

Definition. A family of Borel measures $\{\mu(t, \cdot)\}_{0<t<\infty}$ on $\boldsymbol{H}$ is called a weak solution of Problem (I) on $(0, \infty)$ if it satisfies (II) for any test functional $\Phi(\cdot, \cdot)$ with compact support in t.

Using the Galerkin approximation of the Navier-Stokes equation on (M, g), which appears as a characteristic equation of (I), we may construct a weak solution of (I), that is, a solution of (II), by modifying the arguments in Foiaş [1]. Theorem B is essentially given by the Fourier-Stieltjes transform of the measures obtained in Theorem A, combining with a little geometrical consideration. In proving Theorem C, we use the higher order energy inequality (which is local in time) given, for example, in T. Kato [4] or R. Temam [5].

References

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