16. *Modular Pairs in the Lattice of Projections of a von Neumann Algebra*

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1986)

A pair (a, b) of elements of a lattice is called modular (resp. dual-modular), denoted by $(a, b)M$ (resp. $(a, b)M^*$), if

$$(c\vee a)\wedge b=c\vee(a\wedge b) \text{ for every } c\leqq b$$

$$(\text{resp. } (c\wedge a)\vee b=c\wedge(a\vee b) \text{ for every } c\geqq b)$$

(see [3], (1.1)). If A is a von Neumann algebra, then the set $P(A)$ of all projections of A forms an orthomodular lattice ([3], (37.13) and (37.15)). An algebraic characterization of modular pairs of projections is given in Section 38 of [3]. In this paper, we shall give a norm characterization of modular pairs in $P(A)$, by using the result of Bures [1].

Lemma 1. *Let a, b be elements of an orthomodular lattice, and we put $a_0=a-a\wedge b$, $b_0=b-a\wedge b$ (see [3], (36.5)). Then,*

$$(a, b)M \Longleftrightarrow (a_0, b_0)M \Longleftrightarrow (a, b_0)M.$$

Proof. Assume $(a_0, b_0)M$. Since $a_0\leqq(a\wedge b)^\perp$, we have $a_0Ca\wedge b$ by [3], (36.3). Similarly, $b_0Ca\wedge b$. Hence, it follows from [3], (36.11) that $(a_0\vee(a\wedge b), b_0\vee(a\wedge b))M$, which implies $(a, b)M$.

Next, if we assume $(a, b)M$, then since $aC(a\wedge b)^\perp$, $bC(a\wedge b)^\perp$ and $b\wedge(a\wedge b)^\perp=b_0$, we have $(a, b_0)M$ by [3], (36.11). Finally, if we assume $(a, b_0)M$, then since $a\wedge b_0=(a\wedge b)\wedge b_0\leqq(a\wedge b)\wedge(a\wedge b)^\perp=0$, we have $(a_0, b_0)M$ by [3], (1.5.3).

In [1], two elements a and b of a lattice are said to be modularly separated, if $a\wedge b=0$ and $(a', b')M^*$, $(b', a')M^*$ for all $a'\leqq a$ and $b'\leqq b$.

Lemma 2. *Let A be a von Neumann algebra and let $e, f\in P(A)$.*

(i) $(e, f)M \Longleftrightarrow (f, e)M \Longleftrightarrow (e, f)M^* \Longleftrightarrow (f, e)M^*$.

(ii) *e and f are modularly separated if and only if $e\wedge f=0$ and $(e, f)M$.*

Proof. Since A is a Baer *-ring satisfying the condition (SR) ([3], (37.15)), (i) follows from [3], (29.8) and (37.14). The "only if" part of (ii) follows from (i). Conversely, if $e\wedge f=0$ and $(e, f)M$ then $(e', f')M$ for all $e'\leqq e$, $f'\leqq f$ by [3], (1.5.3). Hence, e and f are modularly separated by (i).

We remark that Theorem 6 and Corollary of Theorem 7 in [1] immediately follow from our Lemma 2 (ii) and [3], (1.6).

Theorem 3. *Let A be a von Neumann algebra and let $e, f\in P(A)$. (e, f) is a modular pair in $P(A)$ if and only if there exist a finite element $e_1\in P(A)$ with $e_1\leqq e-e\wedge f$ and an orthogonal sequence of central projections*

$\{c_n\}$ *with* $\sum_n c_n=1$, *such that*

$$\|(e-e_1)(f-e\wedge f)c_n\|<1 \qquad \text{for all } n.$$

Proof. By Lemmas 1 and 2, (e, f) is modular if and only if $e-e\wedge f$ and $f-e\wedge f$ are modularly separated. So, our theorem immediately follows from [1], Theorem 7, since $(e-e\wedge f-e_1)(f-e\wedge f)=(e-e_1)(f-e\wedge f)$.

Corollary 4. *If* $\|ef-e\wedge f\|<1$ *then* (e, f) *is modular, and the converse is true in case that* A *is a factor of type* III.

Remark 5. If A is a factor of type I, that is, A is the algebra $B(H)$ of all bounded linear operators on a Hilbert space H, then for $e, f \in P(B(H))$ the following statements are equivalent.

(α) (e, f) is modular in $P(B(H))$.

(β) $\|ef-e\wedge f\|<1$.

(γ) $eH+fH$ is closed.

In fact, the equivalence of (α) and (γ) follows from Lemma 2 (i) and [3], (31.10), since $P(B(H))$ is isomorphic to the lattice of all closed subspaces of H by the mapping $e\mapsto eH$. The equivalence of (β) and (γ) follows from [2], Corollary 2.6, since $ef((1-e)\vee(1-f))=ef(1-e\wedge f)=ef-e\wedge f$.

References

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[3] F. Maeda and S. Maeda: Theory of Symmetric Lattices. Springer-Verlag, Berlin (1970).