15. On Diophantine Properties for Convergence of Formal Solutions

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1. Introduction. This paper studies the diophantine properties of the problem of the convergence of formal solutions. Concerning the convergence of formal solutions Kashiwara-Kawai-Sjöstrand studied the equation $Pu \equiv \sum_{|\alpha|=|\beta|\leqq m} a_{\alpha\beta}(x)x^{\alpha}(\partial/\partial x)^{\beta}u(x)=f(x)$ and gave a sufficient condition for the convergence of all formal solutions (cf. [1]). Unfortunately their condition is merely sufficient and not necessary.

As for the necessity there are few works. This is mainly because we must treat rather delicate problems, the diophantine problems. Our object is to study this. We shall introduce the diophantine functions σ_{ξ} and ρ which are the generalizations of the Siegel's condition in [4] and the Leray's auxiliary function in [2] respectively. We note that these authors studied similar problems. In terms of these functions we shall give necessary and sufficient conditions for the convergence of formal solutions. We remark that the method here is also applicable to the study of C^{∞} (or C^{ω})-hypoellipticity of operators on the torus under slight modifications.

2. Notations and results. Let $x=(x_1, x_2)$ be the variable in $\boldsymbol{R}^2$. For $\eta=(\eta_1, \eta_2)\in \boldsymbol{R}^2$ and a multi-index $\alpha=(\alpha_1, \alpha_2)\in \boldsymbol{N}^2$, $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ we set $\eta^{\alpha}=\eta_1^{\alpha_1}\eta_2^{\alpha_2}$ and $(x\cdot\partial)^{\alpha}=(x_1\partial_1)^{\alpha_1}(x_2\partial_2)^{\alpha_2}$ where $\partial=(\partial_1, \partial_2)$ and $\partial_j=\partial/\partial x_j$ $(j=1, 2)$. Let $m\geqq 1$ be an integer and let $\omega\in \boldsymbol{C}$. Then we are concerned with the convergence of all formal solutions of the form $u(x)=x^{\omega}\sum_{\eta\in N^2} u_{\eta}x^{\eta}/\eta!$ of the equation with analytic coefficients:

$$P(x;\partial)u\equiv\sum_{|\alpha|\leqq m} a_{\alpha}(x)\partial^{\alpha}u(x)=f(x)x^{\omega} \tag{2.1}$$

where $f(x)$ is a given analytic function. We say that the formal solution $u=x^{\omega}\sum u_{\eta}x^{\eta}/\eta!$ converges if and only if the sum $\sum u_{\eta}x^{\eta}/\eta!$ converges and represents an analytic function of x. Let us expand $a_{\alpha}(x)$ into Taylor series, $a_{\alpha}(x)=\sum_{\gamma} a_{\alpha,\gamma}x^{\gamma}/\gamma!$ and define the set M_P by $M_P=\{\gamma-\alpha;\ a_{\alpha,\gamma}\neq 0$ for some α and $\gamma\}$. Then we assume:

(A.1) Every $\eta=(\eta_1, \eta_2)$ in M_P satisfies that $\eta_1+\eta_2\geqq 0$ and, if $\eta_1+\eta_2=0$ then either the condition $\eta_1\geqq 0$ or that $\eta_2\geqq 0$ is satisfied for all η in M_P.

Roughly speaking this condition implies that Eq. (2.1) is not irregular-singular. We denote by Γ_P the smallest convex closed cone with its vertex at the origin which contains M_P.

Now let us define

$$p(\eta)=\sum_{\alpha, |\alpha|\leqq m} a_{\alpha,\alpha}\eta!/(\eta-\alpha)!\,\alpha! \tag{2.2}$$

and denote the m-th homogeneous part of $p(\eta)$ by $p_m(\eta)$. For $\xi\in\boldsymbol{R}^2$, $|\xi|=1$ and $\varepsilon>0$ we set $\Gamma(\xi;\varepsilon)=\{\eta\in\boldsymbol{R}^2; |\eta/|\eta|-\xi|<\varepsilon\}$ and define the quantity $\sigma_{\xi,\varepsilon}$ by

$$\sigma_{\xi,\varepsilon}=\sup\{c\in\boldsymbol{R};\ \liminf_{|\eta|\to\infty,\eta\in\Gamma(\xi;\varepsilon)\cap\omega+\boldsymbol{Z}^2}|\eta|^{-c}|p(\eta)|>0\} \tag{2.3}$$

where we put $\sigma_{\xi,\varepsilon}=-\infty$ if $\liminf|\eta|^{-c}|p(\eta)|=0$ for every c in $\boldsymbol{R}$. Note that $\sigma_{\xi,\varepsilon}\leqq m$ since $p(\eta)$ is of degree m. Since $\sigma_{\xi,\varepsilon}$ increases as ε tends to zero we set $\sigma_\xi\equiv\lim_{\varepsilon\downarrow 0}\sigma_{\xi,\varepsilon}$.

Next we define the function ρ following J. Leray [2];

$$\rho=\liminf_{|\eta|\to\infty,\eta\in\boldsymbol{N}^2+\omega}|p(\eta)|^{1/|\eta|}. \tag{2.4}$$

Note that $0\leqq\rho\leqq 1$ because $p(\eta)$ is a polynomial. As for the fundamental properties of σ_ξ and ρ we refer to [5].

We define the differential operator $Q(x;\partial)$ by

$$Q(x;\partial)\equiv\sum_{|\beta|\leqq m_0}b_\beta(x)\partial^\beta=P(x;\partial)-\sum_{|\alpha|\leqq m}a_{\alpha,\alpha}x^\alpha\partial^\alpha/\alpha!$$

where $m_0\leqq m$. We assume the following "quasi-ellipticity condition" on P.

(A.2) For each $\xi\in\Gamma_P$ such that $|\xi|=1$ we have either that $p_m(\xi)\neq 0$ or that $\sigma_\xi>m_0$.

Then our main result is the following.

Theorem 2.1. *Suppose that the conditions* (A.1) *and* (A.2) *are satisfied. Then for every $f(x)$ which is holomorphic in a neighborhood of the origin all formal solutions of Eq.* (2.1) *converge if and only if* $\rho>0$.

Remarks 2.1. a) In Theorem 2.1 we cannot drop any of the assumptions (A.1) and (A.2) in general. Such examples are given in [5].

b) We can generalize Theorem 2.1 to the case of d-independent variables. For the detail we refer to [5].

In Theorem 2.1 we assumed the diophantine conditions $\sigma_\xi>m_0$ and that $\rho>0$. However it is difficult to verify them. The following theorem gives the simple criterion which does not contain diophantine conditions.

Corollary 2.2. *Assume* (A.1) *and suppose that $p_m(\xi)\neq 0$ for all $\xi\in\Gamma_P$. Then, for every $f(x)$ which is holomorphic in a neighborhood of the origin all formal solutions of Eq.* (2.1) *converge.*

We say that the operator P is diophantine-type elliptic if P satisfies (A.2) with Γ_P replaced by $\boldsymbol{R}^2$. The situation is rather simple in this case.

Corollary 2.3. *Suppose that the operator P is diophantine-type elliptic and satisfies* (A.1). *Then we have the same conclusion as in Corollary* 2.2.

Remark 2.2. We can show that the radius of convergence of formal solutions in Corcllary 2.3 is larger than that of $f(x)$ in the right-hand side of (2.1). Hence we can apply Corollary 2.3 to the problem of the holomorphic prolongation of solutions across the characteristic set. Namely we can give diophantine-type sufficient conditions for that this is possible. For the detail we refer to [5].

References

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