112. Groups Associated with Compact Type Subalgebras of Kac-Moody Algebras

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1986)

The Kac-Moody groups associated with a given Kac-Moody algebra as constructed by Peterson-Kac [5] have a disadvantage that the exponential map can not be defined on the whole algebra. The present note gives a partial solution to the problem to remedy the situation, by constructing groups in the above title.

§1. Kac-Moody algebras. Let $\mathfrak{g}$ be a Kac-Moody algebra and A the corresponding generalized Cartan matrix (GCM). Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, Π the set of simple roots, Δ_+ the set of positive roots with respect to Π, and W the Weyl group. We denote by $\mathfrak{g}_{\boldsymbol{R}}$ the Kac-Moody algebra over the real number field $\boldsymbol{R}$ corresponding to the GCM A, and by $\mathfrak{h}_{\boldsymbol{R}}$ the Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$. Then, $\mathfrak{g}=\boldsymbol{C}\otimes\mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{h}=\boldsymbol{C}\otimes\mathfrak{h}_{\boldsymbol{R}}$. There exists an involutive antilinear automorphism ω_0 on $\mathfrak{g}$ such that

(1.1) $$\omega_0(h)=-h \quad (h\in\mathfrak{h}_{\boldsymbol{R}}), \qquad \omega_0(\mathfrak{g}^{\alpha})=\mathfrak{g}^{-\alpha} \quad (\alpha\in\Delta),$$

where $\mathfrak{g}^{\alpha}$ is the α-root space (cf. [3, Chap. 2]). We denote by $\mathfrak{k}$ and $\mathfrak{k}_{\boldsymbol{R}}$ the set of fixed points of ω_0 in $\mathfrak{g}$ and $\mathfrak{g}_{\boldsymbol{R}}$ respectively. Then, $\mathfrak{k}_{\boldsymbol{R}}=\mathfrak{k}\cap\mathfrak{g}_{\boldsymbol{R}}$. Since ω_0 is an involution, $\mathfrak{k}$ is a real form of $\mathfrak{g}$ as a Lie algebra. We call $\mathfrak{k}$ the *unitary form* of $\mathfrak{g}$ and $\mathfrak{k}_{\boldsymbol{R}}$ a *compact type subalgebra* of $\mathfrak{g}_{\boldsymbol{R}}$. If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{g}$ is semisimple, $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$, and $\mathfrak{k}_{\boldsymbol{R}}$ is a maximal compact subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$.

We assume throughout that the GCM A is symmetrizable (cf. [3]). Then, there exists a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$, a standard invariant form, which is infinitesimally invariant under ad $\mathfrak{g}$. The restriction of $(\cdot|\cdot)$ to $\mathfrak{h}$ is W-invariant and non-degenerate, and defines a W-equivariant linear bijection ν from $\mathfrak{h}$ onto its dual $\mathfrak{h}^*$. We denote by the same symbol $(\cdot|\cdot)$ the induced bilinear form on $\mathfrak{h}^*$. Then we have

(1.2) $$[x, y]=(x|y)\nu^{-1}(\alpha) \qquad (x\in\mathfrak{g}^{\alpha},\ y\in\mathfrak{g}^{-\alpha},\ \alpha\in\Delta).$$

We define a sesquilinear form $(\cdot|\cdot)_0$ on $\mathfrak{g}$ as

(1.3) $$(x|y)_0=-(x|\omega_0(y)) \qquad (x, y\in\mathfrak{g}).$$

Then, according to [4, Theorem 1], $(\cdot|\cdot)_0$ is Hermitian and its restriction to each root space $\mathfrak{g}^{\alpha}$ is positive definite.

Put $\mathfrak{n}_{\pm}=\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\pm\alpha}$. Then, they are both subalgebras of $\mathfrak{g}$, and we have a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ (direct sum).

§2. Irreducible highest weight modules. Let $\lambda\in\mathfrak{h}^*$ and L_λ be the left ideal of the enveloping algebra $U(\mathfrak{g})$ generated by $\mathfrak{n}_+$ and $\{h-\lambda(h)\,|\,h\in\mathfrak{h}\}$.

Then, the left $\mathfrak{g}$-module $M(\lambda)=U(\mathfrak{g})/L_\lambda$ is the Verma module for $\mathfrak{g}$ with highest weight λ. We denote by $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$.

Let P be the projection from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ along the decomposition $U(\mathfrak{g})=U(\mathfrak{h})\oplus(U(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- U(\mathfrak{g}))$. Denote by $\langle\cdot|\cdot\rangle_\lambda$ the sesquilinear form on $U(\mathfrak{g})$ defined by

$$\langle x|y\rangle_\lambda = P(y^*x)(\lambda) \qquad (x, y\in U(\mathfrak{g})), \tag{2.1}$$

where we identify $U(\mathfrak{h})$ with the polynomial ring $C[\mathfrak{h}^*]$ on $\mathfrak{h}^*$, and $y\to y^*$ is the unique antilinear antiautomorphism on $U(\mathfrak{g})$ which coincides with $-\omega_0$ on $\mathfrak{g}$. If $\lambda\in\mathfrak{h}^*_R$, then $\langle\cdot|\cdot\rangle_\lambda$ is Hermitian and its restriction to the largest proper ideal of $U(\mathfrak{g})$ containing L_λ is identically zero. Hence, $\langle\cdot|\cdot\rangle_\lambda$ induces a Hermitian form $(\cdot|\cdot)_\lambda$ on $L(\lambda)$. Clearly, $(\cdot|\cdot)_\lambda$ has the following property, called the *contravariance* of $(\cdot|\cdot)_\lambda$.

$$(xu|v)_\lambda = (u|x^*v)_\lambda \qquad (u, v\in L(\lambda),\ x\in\mathfrak{g}). \tag{2.2}$$

Since the GCM A is assumed to be symmetrizable, $(\cdot|\cdot)_\lambda$ is positive definite if λ is dominant integral ([4, Theorem 1]).

§3. Construction of groups associated with the unitary form $\mathfrak{k}$. In this section, we assume that Λ is a dominant integral element of $\mathfrak{h}^*_R$. Let μ be a weight of $L(\Lambda)$ and ρ an element of $\mathfrak{h}^*_R$ taking the value 1 on each simple coroot. From the proof of positivity of $(\cdot|\cdot)_\Lambda$ in [4, Theorem 1], we get the following evaluation of the norm of $\mathfrak{n}_+$-action,

$$\|xv\|^2_\Lambda \leqq 2^{-1}(|\Lambda+\rho|^2-|\mu+\rho|^2)\|x\|^2_0\|v\|^2_\Lambda, \qquad (x\in\mathfrak{n}_+) \tag{3.1}$$

for any element v of the weight space $L(\Lambda)_\mu$, where, for $\lambda\in\mathfrak{h}^*$, $|\lambda|^2=(\lambda|\lambda)$. Making use of this inequality together with the formula (1.2), we obtain an evaluation for the $\mathfrak{n}_-$-action (this time depending on the root α) as

$$\|xv\|^2_\Lambda \leqq 2^{-1}(|\Lambda+\rho|^2-|\mu+\rho|^2+2(\lambda|\alpha))\|x\|^2_0\|v\|^2_\Lambda \tag{3.1$'$}$$

for $x\in\mathfrak{g}^{-\alpha}$, $\alpha\in\Delta_+$, $v\in L(\Lambda)_\mu$. From these evaluations, we have

Theorem 3.1. *For any $0<\varepsilon<1$, there exists an absorbing, symmetric and *-invariant subset B_ε of $\mathfrak{g}$ such that for any $v\in L(\Lambda)$ there exists a positive constant $C=C_\varepsilon$ such that for any $x\in B_\varepsilon$, we have*

$$\|x^m v\|_\Lambda \leqq Cm!\,\varepsilon^m \qquad (m=0, 1, 2, \cdots). \tag{3.2}$$

Hence, the series $\sum_{m=0}^\infty\|(m!)^{-1}x^m v\|_\Lambda$ converges uniformly and is bounded on B_ε.

Let $H(\Lambda)$ be the completion of the pre-Hilbert space $(L(\Lambda), (\cdot|\cdot)_\Lambda)$ and $B=\bigcup_{0<\varepsilon<1}B_\varepsilon$. For any $x\in B$, because of Theorem 3.1, we can define a linear map $\exp x$ from $L(\Lambda)$ into $H(\Lambda)$ by

$$(\exp x)v=\sum_{m=0}^\infty (m!)^{-1}x^m v \qquad (v\in L(\Lambda)). \tag{3.3}$$

By the contravariance (2.2) of $(\cdot|\cdot)_\Lambda$, each element of $\mathfrak{k}$ acts on $L(\Lambda)$ as an antisymmetric operator. Hence, if $x\in B\cap\mathfrak{k}$, then $\exp x$ is an isometry. More strongly,

Proposition 3.2. i) *For any $x\in B\cap\mathfrak{k}$,* $\exp x$ *is uniquely extended to a unitary operator on $H(\Lambda)$, and we have*

$$(\exp x)^{-1}=\exp(-x). \tag{3.4}$$

ii) *If two elements x and y in $B\cap\mathfrak{k}$ commute with each other, then* $\exp x$ *and* $\exp y$ *also commute. If $x+y\in B\cap\mathfrak{k}$ in addition, then we have*

(3.5) $$(\exp x)(\exp y) = \exp(x+y).$$

Let $U(\Lambda)$ be the group of unitary operators on $H(\Lambda)$ equipped with the strong operator topology. By Proposition 3.2, the map exp from $B \cap \mathfrak{k}$ into $U(\Lambda)$ is naturally extended to the whole $\mathfrak{k}$. Let K^Λ be the closed subgroup of $U(\Lambda)$ generated by exp $\mathfrak{k}$, and $H^\Lambda = \exp \sqrt{-1}\mathfrak{h}_R$.

Definition 3.3. We call K^Λ the compact type group associated with the unitary form $\mathfrak{k}$, and H^Λ the Cartan subgroup of K^Λ.

If $\mathfrak{g}$ is finite-dimensional, then K^Λ is a compact Lie group with Lie algebra $\mathfrak{k}$ and H^Λ is a maximal torus of K^Λ. Even when $\mathfrak{g}$ is infinite-dimensional, H^Λ is compact in many cases as follows.

Theorem 3.4. *Let $\Xi(\Lambda)$ be the subgroup of $\mathfrak{h}^*$ generated by all the weights of $L(\Lambda)$. If $\Xi(\Lambda)$ is discrete, then H^Λ is compact and the Pontrjagin dual of H^Λ is isomorphic to $\Xi(\Lambda)$.*

For instance, if $\mathfrak{g}$ is of affine type or the GCM A is non-degenerate, then $\Xi(\Lambda)$ is always discrete (cf. [1]).

The map $\exp : \mathfrak{k} \to K^\Lambda$ is differentiable in the following sense.

Theorem 3.5. *Let $x \in \mathfrak{k}$ and $v \in L(\Lambda)$. Then we have*

(3.6) $$(d/dt)((\exp tx)v) = (\exp tx)(xv) \qquad (t \in \boldsymbol{R}).$$

In other words, every vector of $L(\Lambda)$ is differentiable. By this theorem, the differential of the natural action of K^Λ on $H(\Lambda)$ is the original action of $\mathfrak{k}$ on $L(\Lambda)$, and so we have

Theorem 3.6. *The natural action of K^Λ on $H(\Lambda)$ is irreducible.*

§ 4. Group K_R^Λ associated with $\mathfrak{k}_R$. Let $\mathfrak{p}_R$ be the (-1)-eigenspace of ω_0 in $\mathfrak{g}_R$. We list some facts about $\mathfrak{k}_R$, similar to those in the finite-dimensional case.

i) The restriction of $(\cdot|\cdot)_0$ to $\mathfrak{k}_R$ is positive definite, and so the standard invariant form $(\cdot|\cdot)$ is negative definite on $\mathfrak{k}_R$.

Indeed, $x \in \mathfrak{k}_R$ is written as $x = h + \sum_{\alpha\in\Delta} x_\alpha$ with $h \in \mathfrak{h}_R$, $x_\alpha \in \mathfrak{g}^\alpha$, and

$$x = 2^{-1}(x+\omega_0(x)) = 2^{-1}(h - h + \sum_{\alpha\in\Delta}(x_\alpha + \omega_0(x_\alpha))) = \sum_{\alpha\in\Delta}(x_\alpha + \omega_0(x_\alpha)).$$

Since, $\omega_0(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$, the right hand side of the above equality belongs to $\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$, on which $(\cdot|\cdot)_0$ is positive definite.

ii) $\mathfrak{k}$ is equal to the sum of $\mathfrak{k}_R$ and $\sqrt{-1}\mathfrak{p}_R$.

iii) $\mathfrak{k}$ is generated by $\mathfrak{k}_R$ and $\sqrt{-1}\mathfrak{h}_R$.

Let K_R^Λ be the closed subgroup of K^Λ generated by exp $\mathfrak{k}_R$.

Definition 4.1. We call K_R^Λ the compact type group associated with $\mathfrak{k}_R$.

Remark. If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{k}_R$ is a compact Lie algebra and its complexification is a semisimple Lie algebra, and so a Kac-Moody algebra. But in the infinite-dimensional case, $\boldsymbol{C}\otimes\mathfrak{k}_R = \mathfrak{k}_R + \sqrt{-1}\mathfrak{k}_R$ is not likely to be a Kac-Moody algebra, since $(\cdot|\cdot)_0$ is positive definite on it.

§ 5. Relations with the groups constructed on lowest weight modules. Let $\lambda \in \mathfrak{h}^*$. We denote by L_λ^* the left ideal of $U(\mathfrak{g})$ generated by $\mathfrak{n}_-$ and $\{h + \lambda(h) \mid h \in \mathfrak{h}\}$, and put $M^*(\lambda) = U(\mathfrak{g})/L_\lambda^*$. Then, $M^*(\lambda)$ is the lowest weight Verma module with lowest weight $-\lambda$. Let $L^*(\lambda)$ be the unique irreducible quotient of $M^*(\lambda)$. Denote by P^* the projection from $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ along the decomposition $U(\mathfrak{g})=U(\mathfrak{h})\oplus(U(\mathfrak{g})\mathfrak{n}_- + \mathfrak{n}_+U(\mathfrak{g}))$. By the same argument as in the case of $\langle\cdot|\cdot\rangle_\lambda$, if $\lambda\in\mathfrak{h}^*_R$, we see that the sesquilinear form

$$\langle x|y\rangle^*_\lambda = P^*(y^*x)(-\lambda) \qquad (x, y\in U(\mathfrak{g})) \tag{5.1}$$

is Hermitian and induces a non-degenerate contravariant Hermitian form $(\cdot|\cdot)^*_\lambda$ on $L^*(\lambda)$.

We denote by the same symbol ω_0 the unique antilinear automorphism on $U(\mathfrak{g})$ induced by ω_0 on $\mathfrak{g}$. For $\lambda\in\mathfrak{h}^*_R$, it is clear that $\omega_0(L_\lambda)=L^*_\lambda$, and that the image of any left ideal of $U(\mathfrak{g})$ under w_0 is also a left ideal of $U(\mathfrak{g})$. Hence, ω_0 induces an antilinear bijection Ω_0 from $L(\lambda)$ onto $L^*(\lambda)$. Clearly Ω_0 satisfies

$$\Omega_0(xv)=\omega_0(x)\Omega_0(v) \qquad (x\in\mathfrak{g},\ v\in L(\lambda)). \tag{5.2}$$

In particular, Ω_0 is $\mathfrak{k}$-equivariant. Furthermore, we obtain

Theorem 5.1. *If* $\lambda\in\mathfrak{h}^*_R$, *we have*

$$(\Omega_0(u)|\Omega_0(v))^*_\lambda=(v|u)_\lambda \qquad (u, v\in L(\lambda)). \tag{5.3}$$

Corollary 5.2. *If* $\Lambda\in\mathfrak{h}^*_R$ *is dominant integral, then* $(\cdot|\cdot)^*_\Lambda$ *is positive definite.*

Consider the case $\lambda=\Lambda$ is a dominant integral element of $\mathfrak{h}^*_R$. Let $H^*(\Lambda)$ be the completion of pre-Hilbert space $(L^*(\Lambda), (\cdot|\cdot)^*_\Lambda)$. We can construct a group associated with $\mathfrak{k}$ on $H^*(\Lambda)$ in the same way as in § 3. By Theorem 5.1 and (5.2), we see that this group is isomorphic to K^Λ and that if we identify these groups through antilinear $\mathfrak{k}$-equivariance Ω_0, then the action of K^Λ on $H^*(\Lambda)$ is the contragradient of that on $H(\Lambda)$. Thus,

Theorem 5.3. *Let* $\Lambda\in\mathfrak{h}^*_R$ *be dominant integral. Then,* K^Λ *is represented unitarily and faithfully on* $H^*(\Lambda)$. *This representation is equivalent to the contagradient of the natural representation on* $H(\Lambda)$.

Added in Proof. Recently, a similar evaluation as (3.1) and (3.1′) is given by Mr. E. R. Carrington of Rutgers University. He kindly sent me a handwritten manuscript (without title), and I am grateful to him.

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