106. *Large Time Behavior of a Solution of a Parabolic Equation*

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In this paper, we shall prove that a solution of the following Cauchy problem converges to a constant as $t\to\infty$.

(1) $\partial_t u = Au + \sum_{|\alpha|=2q} B_\alpha(t,x)\partial^\alpha u, \quad t>0, \quad x\in \boldsymbol{R}^d; \quad u(0,x)=u_0(x)$,

where

$$A \equiv (-1)^{q-1}\rho \sum_{k=1}^d \frac{\partial^{2q}}{\partial x_k^{2q}}$$

with a natural number q and a complex number ρ such that $\mathrm{Re}\,\rho>0$, $B_\alpha(t,x)$'s are in a class $\mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ and "smaller" than $\mathrm{Re}\,\rho$, and $u_0(x)$ is in a class $\mathcal{F}^0(\boldsymbol{R}^d)$.

In case of the second order uniformly parabolic equation of the divergence structure, i.e. $\partial_t u = \sum_{j,k=1}^d \partial/\partial x_j (A_{jk}(t,x)\partial u/\partial x_k)$, many authors studied the behavior of the solution as $t\to\infty$ with the order of the convergence (for example see [1, 2]). However their proofs can not be applied to (1), and also in our case u_0 is not necessarily a function in $L_1(\boldsymbol{R}^d)$. Hence our assertion is proved based on the representation of the solution by *a Girsanov type formula* established in [3, 4].

1. For multi index α and $x\in\boldsymbol{R}^d$, we put

$$x^\alpha \equiv \prod_{k=1}^d x_k^{\alpha_k} \quad \text{and} \quad \partial^\alpha \equiv \prod_{k=1} \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

Give a non-negative number κ. $\mathcal{M}^\kappa(\boldsymbol{R}^d)$ is a Banach space consisting of all complex valued measures $\mu(d\xi)$ on $\boldsymbol{R}^d$ with

$$\|\mu\|_\kappa \equiv \int (1+|\xi|)^\kappa |\mu|(d\xi) < \infty,$$

and $\mathcal{F}^\kappa(\boldsymbol{R}^d)$ is a Banach space of all Fourier transforms of $\mathcal{M}^\kappa(\boldsymbol{R}^d)$, i.e.

$$f(x) = \int \exp\{i\xi\cdot x\}\mu_f(d\xi), \qquad \mu_f \in \mathcal{M}^\kappa(\boldsymbol{R}^d),$$

and we define as $\|f\|_\kappa \equiv \|\mu_f\|_\kappa$. $\mathcal{F}^0(\boldsymbol{R}^d)$ is a subset of uniformly continuous and bounded functions, $\sup_x |f(x)| \leqq \|f\|_0$, and the Schwartz class, $\sin\eta\cdot x$, constants, etc. are contained in $\mathcal{F}^\kappa(\boldsymbol{R}^d)$ for any $\kappa \geqq 0$.

Put $\boldsymbol{R}^+ \equiv [0,\infty)$, and $\mathcal{M}^\kappa(\boldsymbol{R}^+, \boldsymbol{R}^d)$ denotes a set of all complex valued measures $\mu(t, d\xi)$, $t\in\boldsymbol{R}^+$, such that (i) $\mu\in\mathcal{M}^\kappa(\boldsymbol{R}^d)$ for each $t\in\boldsymbol{R}^+$, and (ii) $\|\mu(t,\cdot)-\mu(s,\cdot)\|_\kappa \to 0$ as $t\to s$ on $\boldsymbol{R}^+$. $\mathcal{F}^\kappa(\boldsymbol{R}^+, \boldsymbol{R}^d)$ is a Banach space consisting of all Fourier transforms of $\mathcal{M}^\kappa(\boldsymbol{R}^+, \boldsymbol{R}^d)$, i.e.

$$g(t,x) = \int \exp\{i\xi\cdot x\}\mu_g(t, d\xi), \qquad \mu_g \in \mathcal{M}^\kappa(\boldsymbol{R}^+, \boldsymbol{R}^d),$$

with a norm $\sup_{t\geqq 0} \|\mu_g(t,\cdot)\|_\kappa$. $\mu_g^* \in \mathcal{M}^\kappa(\boldsymbol{R}^d)$ is said *a dominating measure* of

$\mu_g \in \mathcal{M}^\kappa(\boldsymbol{R}^+, \boldsymbol{R}^d)$, if

$$\mu_g^*(E) \geqq \sup_{t\geqq 0} \int_E |\mu_g|(t, d\xi)$$

for any Borel set $E \subseteqq \boldsymbol{R}^d$. The Fourier transform $g^*(x)$ *of this* μ_g^* is called *a dominating function* for

$$g(t, x) \equiv \int \exp\{i\xi \cdot x\} \mu_g(t, d\xi).$$

2. Definition. $v(t, x) \in \mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ is *a wide sense solution* of (1), if there is a sequence of sets $\{v^{(m)}(t, x), u_0^{(m)}(x)\}$, $m \geqq 1$, in $\mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d) \times \mathcal{F}^{2q}(\boldsymbol{R}^d)$ which satisfies; (i) for each m, $\partial_t v^{(m)} \in \mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ and $v^{(m)} \in \mathcal{F}^{2q}(\boldsymbol{R}^+, \boldsymbol{R}^d)$, (ii) $v^{(m)}$ is a classical solution of (1) with $u_0 = u_0^{(m)}$, and (iii) $\lim_{m\to\infty} \|u_0^{(m)} - u_0\|_0 = 0$ and $\lim_{m\to\infty} \sup_{t\geqq 0} \|v^{(m)}(t, \cdot) - v(t, \cdot)\|_0 = 0$.

Proposition. *If* $u_0 \in \mathcal{F}^0(\boldsymbol{R}^d)$, *and if each* $B_\alpha \in \mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ *has a dominating function* $B_\alpha^* \in \mathcal{F}^0(\boldsymbol{R}^d)$ *such that* $\sum_{|\alpha|=2q} \|B_\alpha^*\|_0 < \mathrm{Re}\,\rho$, *then there exists a unique wide sense solution of* (1).

The proposition is proved by a little modification of the argument in [4], and the solution is represented by using the generalized Girsanov density. For points $y, \zeta, \xi^{(1)}, \cdots$ in $\boldsymbol{R}^d$, set

$$\langle y \rangle \equiv \left(\textstyle\sum_{k=1}^d y_k^{2q}\right)^{1/2q},$$

$$H(1) \equiv \zeta \quad \text{and} \quad H(l) \equiv \zeta + \xi^{(1)} + \cdots + \xi^{(l-1)} \qquad \text{if } l \geqq 2.$$

We denote by $\mu_0(d\zeta)$ and $\nu_\alpha(t, d\xi)$ the measures corresponding to $u_0(x)$ and $B_\alpha(t, x)$, respectively. From a similar calculation as in [4], we can also write the solution $u(t, x)$ of (1) as

$$u(t, x) = \int \mu_0(d\zeta) \exp\{i\zeta \cdot x - \rho \langle \zeta \rangle^{2q} t\} + \sum_{n=1}^\infty \sum_{|\alpha^{(1)}|=2q} \cdots \sum_{|\alpha^{(n)}|=2q} I(t, x; \alpha^{(1)}, \cdots, \alpha^{(n)}), \tag{2}$$

where, with the convention $s_0 \equiv t$,

$$\begin{aligned} I(t, x; \alpha^{(1)}, \cdots, \alpha^{(n)}) \equiv & \int_{t>s_1>\cdots>s_n>0} ds_1 \cdots ds_n \int \mu_0(d\zeta) \\ & \times \int \nu_{\alpha^{(1)}}(t-s_1, d\xi^{(1)}) \cdots \int \nu_{\alpha^{(n)}}(t-s_n, d\xi^{(n)}) \exp\{iH(n+1)\cdot x\} \\ & \times \left[\textstyle\prod_{l=1}^n (iH(l))^{\alpha^{(l)}} \exp\{-\rho \langle H(l) \rangle^{2q} (s_{l-1} - s_l)\}\right] \\ & \times \exp\{-\rho \langle H(n+1) \rangle^{2q} s_n\}. \end{aligned} \tag{3}$$

3. Our assertion in this paper is:

Theorem. *Under the assumptions in the proposition,* $u(t, x)$ *converges to a constant in* $\|\ \|_0$ *sense as* $t \to \infty$.

Corollary. *If the measure* μ_0 *corresponding to* u_0 *is absolutely continuous in the Lebesgue measure, i.e.*

$$\mu_0(d\zeta) = \hat{u}_0(\zeta) d\zeta \qquad \textit{for } \hat{u}_0 \in L_1(\boldsymbol{R}^d),$$

then the constant in the theorem is zero.

4. Using (2) and (3), we shall prove the theorem and the corollary. Let $\{u^{(m)}(t, x), u_0^{(m)}(x)\}$ be a sequence as in the definition.

Step 1. First, we show that $u^{(m)}$ converges to a constant in $\|\ \|_0$ sense as $t \to \infty$, for each m.

Denote by $\mu_0^{(m)}$ the corresponding measure to $u_0^{(m)}$, and put $\theta \equiv$

$\sum_{|\alpha|=2q}\|B^*_\alpha\|_0/\mathrm{Re}\,\rho$. Since $|y^\alpha|\leqq\langle y\rangle^{2q}$ for $|\alpha|=2q$,

$$\int_0^\infty \|\partial^\beta I^{(m)}(t,\cdot\,;\alpha^{(1)},\cdots,\alpha^{(n)})\|_0\,dt\leqq\frac{\|u_0^{(m)}\|_0}{(\mathrm{Re}\,\rho)^{n+1}}\|B^*_{\alpha^{(1)}}\|_0\cdots\|B^*_{\alpha^{(n)}}\|_0$$

for $|\beta|=2q$, where $I^{(m)}$ is defined on (3) with $\mu_0^{(m)}$ in the place of μ_0. By this and (2),

$$\int_0^\infty \|\partial^\beta u^{(m)}(t,\cdot)\|_0\,dt\leqq\frac{\|u_0^{(m)}\|_0}{(\mathrm{Re}\,\rho)(1-\theta)}\qquad\text{for } |\beta|=2q. \tag{4}$$

After a similar calculation as above, we observe;

$$\int_0^\infty \|\partial_t u^{(m)}(t,\cdot)\|_0\,dt\leqq\frac{(1+|\rho|)\|u_0^{(m)}\|_0}{(\mathrm{Re}\,\rho)(1-\theta)}, \tag{5}$$

$$\sup_{t\geqq 0}\|u^{(m)}(t,\cdot)\|_0\leqq\frac{\|u_0^{(m)}\|_0}{1-\theta}. \tag{6}$$

From (4), we can take a sequence $\{t_p\}$ tending to infinity, and

$$\lim_{p\to\infty}\|\partial^\beta u^{(m)}(t_p,\cdot)\|_0=0\qquad\text{for } |\beta|=2q.$$

On the Taylor expansion

$$u^{(m)}(t_p,x)-u^{(m)}(t_p,0)=\sum_{1\leqq|\beta|\leqq 2q-1}\frac{x^\beta}{|\beta|!}\partial^\beta u^{(m)}(t_p,0)$$
$$+\sum_{|\beta|=2q}\frac{x^\beta}{|\beta|!}\partial^\beta u^{(m)}(t_p,y^{(p)}),\qquad 0\leqq|y^{(p)}|\leqq|x|,$$

we let $p\to\infty$, then (6) and the fact as stated above yield

$$\overline{\lim}_{p\to\infty}\left|\sum_{1\leqq|\beta|\leqq 2q-1}\frac{x^\beta}{|\beta|!}\partial^\beta u^{(m)}(t_p,0)\right|\leqq\frac{2\|u_0^{(m)}\|_0}{1-\theta}.$$

This requires that $\lim_{p\to\infty}\partial^\beta u^{(m)}(t_p,0)=0$ for $1\leqq|\beta|\leqq 2q-1$, because x may be large enough. Consequently, we obtain

$$\lim_{p\to\infty}u^{(m)}(t_p,x)=\lim_{p\to\infty}u^{(m)}(t_p,0)\equiv c_\infty^{(m)}. \tag{7}$$

On the other hand, since (5) derives that

$$\|u^{(m)}(T,\cdot)-u^{(m)}(T',\cdot)\|_0\leqq\int_{T'}^{T}\|\partial_t u^{(m)}(t,\cdot)\|_0\,dt\to 0$$

as $T,T'\to\infty$, $u^{(m)}(t,x)$ converges in $\|\ \|_0$ sense as $t\to\infty$. Combine this with (7), then it follows that $\lim_{t\to\infty}\|u^{(m)}(t,\cdot)-c_\infty^{(m)}\|_0=0$ for each m.

Step 2. From (6), we see that $\{c_\infty^{(m)}\}_{m\geqq 1}$ is a Cauchy sequence, and set $c_\infty\equiv\lim_{m\to\infty}c_\infty^{(m)}$. Notice that

$$\sup_{t>T}\|u(t,\cdot)-c_\infty\|_0\leqq\sup_{t>T}\|u(t,\cdot)-u^{(m)}(t,\cdot)\|_0$$
$$+\sup_{t>T}\|u^{(m)}(t,\cdot)-c_\infty^{(m)}\|_0+|c_\infty^{(m)}-c_\infty|,$$

and the theorem follows.

Step 3. Due to the assumption in the corollary, we can apply Riemann-Lebesgue's theorem to (2), and get

$$\lim_{|x|\to\infty}u(t,x)=0\qquad\text{for each } t\geqq 0.$$

Now a combination of this and the theorem derives the corollary.

References

[1] Eskin, L. D.: On the asymptotic behavior of solutions of parabolic equations as $t\to\infty$. Izv. Vyss. Ucebn. Zaved. Matematika, **54**, 154–164 (1966) (Amer. Math.

Soc. Transl., **87**, 89–98 (1970)).
[2] Nash, J.: Continuity of solutions of parabolic and elliptic equations. Amer. J. Math., **80**, 931–954 (1958).
[3] Nishioka, K.: Stochastic calculus for a class of evolution equations. Japan. J. Math., **11**, 59–102 (1985).
[4] ——: A stochastic solution of a high order parabolic equation (to appear in J. Math. Soc. Japan).