105. The 102-Velocity Model and the Related Discrete Models of the Boltzmann Equation

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In the preceding paper [1], we established the existence of infinitely many regular discrete models invariant under the transformation group $G=A_5\times I$, a finite subgroup of $O(3)$. All the models constructed in [1] have 8 linearly independent summational invariants. In this note, we show the following: There exist also infinitely many regular discrete models with the symmetry group $G=A_5\times I$ such that the associated spaces of summational invariants have 9 dimensions. All these models are normal in the sense that any summational invariant can be expressed in terms of the summational invariants issuing from the conservation laws of mass, momentum, and energy. Note that, so far as the models are normal and described by means of a certain quadratic field, the maximal number of the dimension of the space of summational invariants is 9. We study first the 102-velocity model defined by combining the 12-velocity model with the 90-velocity model. Then we construct a sequence of regular discrete models by induction.

1. We recall that the 90-velocity model is defined by using a dodecahedron with its center at the origin. Let $\tau=(1+\sqrt{5})/2$. Then, $\{1,\tau\}$ forms the standard basis of the quadratic field $Q(\sqrt{5})$. By means of this basis, the coordinates of $v_1,\cdots,v_{90}$ are expressed by sextuples of rational integers as follows.

$v_1=(0,2,0,0,0,0)$, $v_2=(0,0,0,2,0,0)$, $v_3=(0,0,0,0,0,2)$,
$v_7=(1,0,1,1,0,-1)$, $v_8=(-1,-1,0,1,1,0)$, $v_9=(0,1,1,0,1,1)$,
$v_{13}=(-1,0,1,1,0,-1)$, $v_{14}=(-1,-1,0,-1,-1,0)$, $v_{15}=(0,-1,1,0,1,1)$,
$v_{19}=(-1,0,-1,-1,0,-1)$, $v_{20}=(1,1,0,-1,1,0)$, $v_{21}=(0,-1,-1,0,1,1)$,
$v_{25}=(1,0,-1,-1,0,-1)$, $v_{26}=(1,1,0,1,-1,0)$, $v_{27}=(0,1,-1,0,1,1)$,
$v_i=-v_{i-3}$ for $i=j+6k-3$, $1\leq j\leq 3$, $1\leq k\leq 5$, $v_{31}=(0,2,0,0,0,2)$,
$v_{32}=(0,0,0,2,0,2)$, $v_{33}=(0,-2,0,0,0,2)$, $v_{34}=(0,0,0,-2,0,2)$,
$v_{39}=(0,2,0,2,0,0)$, $v_{40}=(0,-2,0,2,0,0)$, $v_{43}=(1,1,2,1,1,0)$,
$v_{44}=(-1,0,1,1,2,1)$, $v_{45}=(-1,1,0,-1,1,2)$, $v_{46}=(1,2,1,-1,0,1)$,
$v_{51}=(0,-1,1,2,1,-1)$, $v_{52}=(-2,-1,-1,0,1,1)$, $v_{55}=(-1,-1,2,1,1,0)$,
$v_{56}=(-1,-2,1,-1,0,1)$, $v_{57}=(1,-1,0,-1,1,2)$, $v_{58}=(1,0,1,1,2,1)$,
$v_{63}=(-2,-1,1,0,-1,-1)$, $v_{64}=(0,-1,-1,-2,-1,1)$,

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$v_{67}=(-1,-1,-2,-1,1,0)$, $v_{68}=(1,0,-1,-1,2,1)$, $v_{69}=(1,-1,0,1,1,2)$,
$v_{70}=(-1,-2,-1,1,0,1)$, $v_{75}=(0,1,-1,-2,1,-1)$, $v_{76}=(2,1,1,0,1,1)$,
$v_{79}=(1,1,-2,-1,1,0)$, $v_{80}=(1,2,-1,1,0,1)$, $v_{81}=(-1,1,0,1,1,2)$,
$v_{82}=(-1,0,-1,-1,2,1)$, $v_{87}=(2,1,-1,0,-1,-1)$, $v_{88}=(0,1,1,2,-1,1)$,
$v_i=-v_{i-4}$ for $i=j+12k+22$, $1\leq j\leq 4$, $1\leq k\leq 5$,
$v_i=-v_{i-2}$ for $i=j+12k+28$, $j=1,2$, $1\leq k\leq 5$.

We add further 12 velocities given below.
$v_{91}=(0,0,2,0,0,2)$, $v_{92}=(0,0,-2,0,0,2)$, $v_{95}=(2,0,0,2,0,0)$,
$v_{96}=(-1,0,0,2,0,0)$, $v_{99}=(0,2,0,0,2,0)$, $v_{100}=(0,2,0,0,-2,0)$,
$v_i=-v_{i-2}$ for $i=j+4k+88$, $j=1,2$, $1\leq k\leq 3$.

We set $M_1=\{v_1,\cdots,v_{30}\}$, $M_2=\{v_{31},\cdots,v_{90}\}$, $M_3=\{v_{91},\cdots,v_{102}\}$. Then $M=M_1\cup M_2\cup M_3$ is the 102-velocity model. This is a model with three different moduli of velocities. Note that M_3, when combined with the set of 20 vertices of the dodecahedron used for defining the 90-velocity model, forms the 32-velocity model with $\gamma=\gamma_1$. (See [2].)

2. We enumerate the collisions of the 90-velocity model. Let $p^{(1,3)}$ and $p^{(2,3)}$ be the numbers of mixed collisions of M_1, M_3 and M_2, M_3, respectively. We have $p^{(1,3)}=60$, $p^{(2,3)}=240$. The respective numbers of types of collisions are $q^{(1,3)}=1$ and $q^{(2,3)}=3$. It is known that $p^{(3)}$, the number of collisions of M_3, is equal to 30 and that $q^{(3)}$, the number of types of collisions of M_3 is equal to 1. Taking into account of the results obtained in [1], we conclude that p, the total number of collisions of M, is equal to 4,410 and that q, the total number of types of collisions of M, is equal to 49. The results concerning the P-sets are as follows. Let $1\leq i\leq 102$ and let $r^{(i)}$ be the number of P-sets whose cardinalities are equal to i. Then, $r^{(3)}=50,740$, $r^{(4)}=6,390$, $r^{(5)}=1,044$, $r^{(6)}=1,130$, $r^{(7)}=690$, $r^{(8)}=150$, $r^{(9)}=100$, $r^{(10)}=6$, $r^{(12)}=35$, $r^{(15)}=12$, $r^{(18)}=10$, $r^{(20)}=12$. We have $r^{(i)}=0$ for other i's. It follows that, r, the total number of P-sets, equals 60,319. We classify the P-sets by using the symmetry group $G=A_5\times I$. We denote by $s^{(i)}$ the number of classes obtained from the P-sets whose cardinalities are equal to i. Here, $3\leq i\leq 10$ or $i=12,15,18,20$. Then, $s^{(3)}=444$, $s^{(4)}=83$, $s^{(5)}=16$, $s^{(6)}=21$, $s^{(7)}=12$, $s^{(8)}=4$, $s^{(9)}=3$, $s^{(10)}=1$, $s^{(12)}=2$, $s^{(15)}=s^{(18)}=s^{(20)}=1$. Hence, s, the total number of equivalence classes equals 589. The space of the summational invariants has dimension 9 for the 102-velocity model. Finally we check the stability condition in the reduced form by using the computer.

Theorem 1. *The 102-velocity model is a regular discrete model.*

Theorem 2. *Let G be $A_5\times I$ and let l be either* 8 *or* 9. *Let $\mathcal{R}_l(G)$ be the set of regular discrete models whose spaces of the summational invariants have dimension l. Let, furthermore, $k\geq 2$ be an integer. Then, $\mathcal{R}_l(G)$ contains a model with k moduli of velocities.*

The latter theorem is a refinement of Theorem 2 of [1]. The proof is analogous. We give the list of types of collisions below. E and M denote the energy and the modulus of the momentum, respectively. I stands for the modulus of cos θ, where θ is the angle of deflection.

Classification of mixed collisions

type	number of collision	representative	E	M	I
C_1	30	$(v_1, v_4)\to(v_2, v_5)$	$8(1+\tau)$	0	0
C_2	60	$(v_1, v_4)\to(v_7, v_{10})$	$8(1+\tau)$	0	$(-1+\tau)/2$
C_3	60	$(v_1, v_4)\to(v_8, v_{11})$	$8(1+\tau)$	0	$\tau/2$
C_4	60	$(v_1, v_4)\to(v_9, v_{12})$	$8(1+\tau)$	0	$1/2$
C_5	60	$(v_1, v_7)\to(v_{17}, v_{24})$	$8(1+\tau)$	$4(2+3\tau)$	$(-1+2\tau)/5$
C_6	60	$(v_1, v_8)\to(v_{13}, v_{27})$	$8(1+\tau)$	4	$(-1+2\tau)/5$
C_7	60	$(v_1, v_{12})\to(v_{19}, v_{26})$	$8(1+\tau)$	0	$(-1+2\tau)/3$
C_8	30	$(v_{31}, v_{35})\to(v_{33}, v_{37})$	$16(1+\tau)$	0	0
C_9	120	$(v_{31}, v_{35})\to(v_{32}, v_{36})$	$16(1+\tau)$	0	$1/2$
C_{10}	120	$(v_{31}, v_{35})\to(v_{43}, v_{47})$	$16(1+\tau)$	0	$(-1+2\tau)/4$
C_{11}	120	$(v_{31}, v_{35})\to(v_{44}, v_{48})$	$16(1+\tau)$	0	$\tau/4$
C_{12}	60	$(v_{31}, v_{35})\to(v_{45}, v_{49})$	$16(1+\tau)$	0	$3/4$
C_{13}	60	$(v_{31}, v_{35})\to(v_{46}, v_{50})$	$16(1+\tau)$	0	$(2+\tau)/4$
C_{14}	60	$(v_{31}, v_{35})\to(v_{51}, v_{53})$	$16(1+\tau)$	0	$(3-\tau)/4$
C_{15}	120	$(v_{31}, v_{35})\to(v_{52}, v_{54})$	$16(1+\tau)$	0	$(-1+\tau)/4$
C_{16}	60	$(v_{31}, v_{35})\to(v_{55}, v_{59})$	$16(1+\tau)$	0	$1/4$
C_{17}	60	$(v_{31}, v_{35})\to(v_{58}, v_{62})$	$16(1+\tau)$	0	$(-2+3\tau)/4$
C_{18}	60	$(v_{31}, v_{35})\to(v_{63}, v_{65})$	$16(1+\tau)$	0	$(-1+3\tau)/4$
C_{19}	120	$(v_{31}, v_{36})\to(v_{34}, v_{37})$	$16(1+\tau)$	$8(1+\tau)$	$1/3$
C_{20}	60	$(v_{31}, v_{33})\to(v_{32}, v_{34})$	$16(1+\tau)$	$16(1+\tau)$	0
C_{21}	60	$(v_{31}, v_{49})\to(v_{65}, v_{72})$	$16(1+\tau)$	$4(1+\tau)$	$3(-1+2\tau)/7$
C_{22}	60	$(v_{31}, v_{50})\to(v_{67}, v_{88})$	$16(1+\tau)$	4	$(29-10\tau)/41$
C_{23}	60	$(v_{31}, v_{35})\to(v_{55}, v_{80})$	$16(1+\tau)$	$4(2+3\tau)$	$(19+10\tau)/41$
C_{24}	60	$(v_{31}, v_{58})\to(v_{76}, v_{81})$	$16(1+\tau)$	$4(5+8\tau)$	$(-1+2\tau)/3$
C_{25}	60	$(v_{31}, v_{59})\to(v_{53}, v_{80})$	$16(1+\tau)$	$20(1+\tau)$	$(-1+2\tau)/3$
C_{26}	60	$(v_{31}, v_{63})\to(v_{45}, v_{72})$	$16(1+\tau)$	$4(2-\tau)$	$(-1+2\tau)/3$
C_{45}	30	$(v_{91}, v_{93})\to(v_{92}, v_{94})$	$8(2+\tau)$	0	$(-1+2\tau)/5$
C_{46}	60	$(v_1, v_{101})\to(v_4, v_{100})$	$4(3+2\tau)$	4	$(11+8\tau)/29$
C_{47}	120	$(v_{31}, v_{93})\to(v_{37}, v_{92})$	$4(4+3\tau)$	$4(2+\tau)$	$(7+8\tau)/41$
C_{48}	60	$(v_{31}, v_{101})\to(v_{33}, v_{100})$	$4(4+3\tau)$	$4(2-\tau)$	$(19-8\tau)/29$
C_{49}	60	$(v_{31}, v_{102})\to(v_{33}, v_{99})$	$4(4+3\tau)$	$4(2+3\tau)$	$(1+8\tau)/15$

References

[1] Y. Shizuta, M. Maeji, A. Watanabe, and S. Kawashima: Regularity of the 90-velocity model of the Boltzmann equation. Proc. Japan Acad., **62**A, 171–173 (1986).
[2] S. Kawashima, A. Watanabe, M. Maeji, and Y. Shizuta: On Cabannes' 32-velocity models of the Boltzmann equation. Publ. RIMS, Kyoto Univ., **22**, 583–607 (1986).