97. A Note on Heegaard Splittings of Non-orientable Surface Bundles over S^1

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Let F_g be an orientable closed surface with $H_1(F_g, Z)\cong\oplus^{2g} Z$, and N_h be a non-orientable closed surface with $H_1(N_h, Z_2)\cong\oplus^h Z_2$. In [2], Takahashi and Ochiai proved that F_g (N_h resp.)-bundle over S^1 admits a Heegaard splitting of genus $2g+1$ ($h+1$ resp.). Further they proved that for any g (≥ 0) there exists an F_g-bundle over S^1 which admits a Heegaard splitting of genus two (Theorem 2 of [2]). In this note, we will show a similar result for non-orientable surface bundles.

Theorem. *For any h (≥ 1) there exists an N_h-bundle over S^1 which admits a Heegaard splitting of genus two.*

Proof. Case 1. h is odd.

Let E_h be a $(2, h)$-torus knot exterior in S^3. Then E_h is an F_g^1-bundle over S^1, where F_g^1 is an F_g with one hole and $g=(h-1)/2$ (see Ch. 10 of [1]). Let λ be the boundary of a fiber of the fibration of E_h, and μ be a meridian in ∂E_h such that λ intersects μ in a single point. Let B be a Möbius band. Put $L=B\times S^1$, $\alpha=\partial B\times\{a\}$ and $\beta=\{b\}\times S^1$, where $a\in S^1$ and $b\in\partial B$. Let f be a homeomorphism of ∂E_h to ∂L with $f(\lambda)=\alpha$ and $f(\mu)=\beta$. Let M_h be a 3-manifold obtained from E_h and L by identifying ∂E_h and ∂L by f. Then it is clear that M_h is an N_h-bundle over S^1.

Since $(2, h)$-torus knot is a 2-bridge knot, there exists a 2-sphere with four holes S properly embedded in E_h such that each component of ∂S is a meridian. Then S cuts E_h into two genus two orientable handlebodies V_1 and V_2. Let μ_1, μ_2, μ_3 and μ_4 be four components of ∂S and put $f(\mu_i)\cap\alpha=\{x_i\}$ ($i=1, 2, 3, 4$). Then, by changing the letters if necessary, there exist two essential arcs γ and δ in $B\times\{a\}$ with $\partial\gamma=\{x_1, x_2\}$ and $\partial\delta=\{x_2, x_4\}$. Then $(\gamma\cup\delta)\times S^1$ cuts L into two solid tori T_1 and T_2. Then we may assume that $f(\mathrm{Cl}(\partial V_i-S))=\mathrm{Cl}(\partial T_i-(\gamma\cup\delta)\times S^1)$ ($i=1, 2$). Let H_i be a 3-manifold obtained $V_i\cup T_i$ by identifying x with $f(x)$ for any $x\in\mathrm{Cl}(\partial V_i-S)$ ($i=1, 2$), then H_i is a non-orientable genus two handlebody. Therefore $M_h=H_1\cup H_2$ is a Heegaard splitting of genus two of M_h.

Case 2. h is even.

Let E_h be a $(2, h)$-torus link exterior in S^3. Then E_h is an F_g^2-bundle over S^1, where F_g^2 is an F_g with two holes and $g=(h-2)/2$ (see Ch. 10 of [1]). Let λ_1, λ_2 be two components of the boundary of a fiber of the fibration of E_h and μ_1, μ_2 be two meridians in ∂E_h such that λ_i intersects μ_i in a single point ($i=1, 2$). We give orientations to $\lambda_1, \lambda_2, \mu_1$ and μ_2 as follows.

The orientation of μ_i is a fixed direction vertical to a fiber ($i=1,2$). The orientation of λ_i is the direction induced from an orientation of a fiber ($i=1,2$).

Let T_1, T_2 be two components of ∂E_h with $(\lambda_i \cup \mu_i) \subset T_i$ ($i=1,2$). Let f be a homeomorphism of T_1 to T_2 such that $f(\lambda_1)=\lambda_2$, $f(\mu_1)=\mu_2$ and both $f|\lambda_1$ and $f|\mu_1$ are orientation preserving. Let M_h be a 3-manifold obtained from E_h by identifying T_1 and T_2 by f. Then it is clear that M_h is an N_h-bundle over S^1.

Let S be a 2-sphere with four holes in E_h which cuts E_h into two genus two handlebodies V_1 and V_2. Then, by moving f by an isotopy if necessary, we may assume that $f(\mathrm{Cl}(\partial V_i - S) \cap T_1) = \mathrm{Cl}(\partial V_i - S) \cap T_2$ ($i=1,2$). Let H_i be a 3-manifold obtained from V_i by identifying x with $f(x)$ for any $x \in \mathrm{Cl}(\partial V_i - S) \cap T_1$ ($i=1,2$), then H_i is a non-orientable genus two handlebody. Therefore $M_h = H_1 \cup H_2$ is a Heegaard splitting of genus two of M_h.

This completes the proof.

Remark. In Case 2, if we choose f so that $f|\lambda_1$ is orientation reversing and $f|\mu_1$ is orientation preserving, then M_h is an F_{g+1}-bundle over S^1 which admits a Heegaard splitting of genus two. This is an alternative proof of Theorem 2 of [2].

References

[1] D. Rolfsen: Knots and links. Mathematics Lectures Series, **7**, Publish or Perish Inc., Berkeley, Ca. (1976).

[2] M. Takahashi and M. Ochiai: Heegaard diagrams of torus bundles over S^1. Comment. Math. Univ. S. Paul., **31**, 63–69 (1982).