96. *Product of Linear Operators with Closed Range*

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1. Introduction. Let X, Y be normed linear spaces and let T be a linear operator with domain $\mathrm{D}(T)$ in X and range $\mathrm{R}(T)$ in Y. The null space of T is denoted by $\mathrm{N}(T)$. Then the *lower bound* (or *reduced minimum modulus*) of T is defined by

$$\gamma(T)=\sup\{\gamma : \|Tx\|\geqq\gamma \operatorname{dist}(x, \mathrm{N}(T)) \ (x\in \mathrm{D}(T))\}$$

where $\operatorname{dist}(x, \mathrm{N}(T))$ denotes the distance from x to $\mathrm{N}(T)$. If X, Y are Banach spaces and T is a closed linear operator, then it is well known that $\mathrm{R}(T)$ is a closed subspace of Y if and only if $\gamma(T)>0$ (cf. [2]).

Now let Z be another normed linear space and let S be a linear operator from Y to Z. Then the product ST of S and T is defined as a linear operator from X to Z. In [3], an estimate is obtained bounding $\gamma(ST)$ from below in terms of the product of $\gamma(S)$ and $\gamma(T)$. The main purpose of this note is to give the estimate of $\gamma(\hat{S})$, where $\hat{S}$ denotes the restriction of S to $\mathrm{R}(T)$. As a consequence, we can obtain a result of R. Bouldin which gives a necessary and sufficient condition for the product ST to have closed range in case S is a bounded linear operator with $\mathrm{D}(S)=Y$ (cf. [1]).

2. Gap and angular distance between closed subspaces. Let E be a normed linear space and let M, N be *non-trivial closed* subspaces of E. We denote by S_M the set of all $x\in M$ such that $\|x\|=1$. In this section, we consider the following quantities between M and N:

$$\alpha(M, N)=\inf\{\|x-y\| : x\in S_M,\ y\in S_N\},$$
$$\beta(M, N)=\sup\{\beta : \operatorname{dist}(x, N)\geqq\beta\|x\| \ (x\in M)\},$$
$$\gamma(M, N)=\sup\{\gamma : \operatorname{dist}(x, N)\geqq\gamma \operatorname{dist}(x, M\cap N) \ (x\in M)\},$$

and study the relations between them. $\alpha(M, N)$ is called the *angular distance* between M, N; while $\gamma(M, N)$ is called the *gap* between M, N (cf. [1], [2]). For a Banach space E, it is well known that $\gamma(M, N)>0$ if and only if $M+N$ is a closed subspace of E (cf. [2]).

Lemma 1. $\beta(M, N)\leqq\alpha(M, N)\leqq 2\beta(M, N)$.

Proof. Since we have

$$\alpha(M, N)=\inf\{\operatorname{dist}(x, S_N) : x\in S_M\},$$
$$\beta(M, N)=\inf\{\operatorname{dist}(x, N) : x\in S_M\},$$

it is clear that $\beta(M, N)\leqq\alpha(M, N)$. The other inequality follows from the following fact which is proved in [2] on p. 198:

$$\operatorname{dist}(x, S_N)\leqq 2 \operatorname{dist}(x, N)$$

for any $x\in E$ with $\|x\|=1$.

Theorem 2. $\gamma(M, N)\leqq\alpha(M/M\cap N,\ N/M\cap N)\leqq 2\gamma(M, N)$.

Proof. First we consider the special case where $M\cap N=\{0\}$. Then by the above lemma, we have

$$\gamma(M,N)=\beta(M,N)\leqq\alpha(M,N)\leqq 2\beta(M,N)=2\gamma(M,N).$$

In the general case where $M\cap N\neq\{0\}$, we set $E_o=M\cap N$ and consider the quotient space $\tilde{E}=E/E_o$. We denote by $\tilde{u}$ the coset to which u belongs. Since E_o is closed, $\tilde{E}$ is also a normed linear space under the quotient norm. Let $\tilde{M}=M/E_o$ and $\tilde{N}=N/E_o$. Then $\tilde{M}$ and $\tilde{N}$ are closed subspaces of $\tilde{E}$ with $\tilde{M}\cap\tilde{N}=\{\tilde{0}\}$ and it is easily verified that

$$\operatorname{dist}(\tilde{u},\tilde{N})=\operatorname{dist}(u,N),$$
$$\operatorname{dist}(\tilde{u},\tilde{M}\cap\tilde{N})=\|\tilde{u}\|=\operatorname{dist}(u,M\cap N),$$

so that we have $\beta(\tilde{M},\tilde{N})=\gamma(M,N)$. Hence the proof of the general case follows from the above lemma as follows:

$$\gamma(M,N)=\beta(\tilde{M},\tilde{N})\leqq\alpha(\tilde{M},\tilde{N})\leqq 2\beta(\tilde{M},\tilde{N})=2\gamma(M,N).$$

Corollary 3. *Let E be a Banach space and let M, N be closed subspaces of E. Then the following conditions are equivalent:*

(1) *$M+N$ is a closed subspace of E.*

(2) $\gamma(M,N)>0$. (3) $\alpha(M/M\cap N,\ N/M\cap N)>0$.

Remark 4. If E is an inner product space over the complex numbers, then we have the following improved estimate between $\alpha(M,N)$ and $\beta(M,N)$:

$$\beta(M,N)\leqq\alpha(M,N)\leqq\sqrt{2}\,\beta(M,N).$$

This follows from the following relations:

$$[\alpha(M,N)]^2=2[1-\tau(M,N)],\qquad [\beta(M,N)]^2+[\tau(M,N)]^2=1$$

where $\tau(M,N)$ is defined by

$$\tau(M,N)=\sup\{|\langle a,b\rangle|: a\in S_M,\ b\in S_N\}.$$

3. Estimate of the lower bound. Throughout this section, we assume that X, Y, Z are normed linear spaces, T is a linear operator from X to Y, S is a *non-trivial* linear operator from Y to Z and $\mathrm{R}(T), \mathrm{R}(S)$ are *closed* subspaces of Y, Z respectively. Moreover, we denote by $\hat{S}$ the restriction of S to $\mathrm{R}(T)$: $\hat{S}(Tx)=S(Tx)$ ($x\in\mathrm{D}(ST)$).

In this section, we shall prove the following estimate of the lower bound of $\hat{S}$.

Theorem 5. (1) $\gamma(\hat{S})\geqq\gamma(S)\gamma(\mathrm{R}(T),\mathrm{N}(S))$.

(2) *If S is a bounded linear operator with $\mathrm{D}(S)=Y$, then we have:*

$$\|S\|\,\gamma(\mathrm{R}(T),\mathrm{N}(S))\geqq\gamma(\hat{S}).$$

Proof. Since $\mathrm{N}(\hat{S})=\mathrm{N}(S)\cap\mathrm{R}(T)$, we have

$$\begin{aligned}\|\hat{S}(Tx)\|=\|S(Tx)\|&\geqq\gamma(S)\operatorname{dist}(Tx,\mathrm{N}(S))\\&\geqq\gamma(S)\gamma(\mathrm{R}(T),\mathrm{N}(S))\operatorname{dist}(Tx,\mathrm{N}(S)\cap\mathrm{R}(T))\\&=\gamma(S)\gamma(\mathrm{R}(T),\mathrm{N}(S))\operatorname{dist}(Tx,\mathrm{N}(\hat{S}))\end{aligned}$$

for each $x\in\mathrm{D}(ST)$. Hence we get $\gamma(\hat{S})\geqq\gamma(S)\gamma(\mathrm{R}(T),\mathrm{N}(S))$.

Now assume that S is a bounded operator with $\mathrm{D}(S)=Y$ and let $x\in\mathrm{D}(ST)=\mathrm{D}(T)$. Then for any $y\in\mathrm{N}(S)$, we have

$$\|STx\|=\|S(Tx-y)\|\leqq\|S\|\,\|Tx-y\|$$

and hence

$$\|STx\|\leqq\|S\|\operatorname{dist}(Tx,\mathrm{N}(S)).$$

On the other hand, we also have

$$\|STx\| = \|\hat{S}(Tx)\| \geqq \gamma(\hat{S}) \operatorname{dist}(Tx, \mathrm{N}(\hat{S}))$$
$$= \gamma(\hat{S}) \operatorname{dist}(Tx, \mathrm{N}(S) \cap \mathrm{R}(T)).$$

Therefore we get

$$\|S\| \operatorname{dist}(Tx, \mathrm{N}(S)) \geqq \gamma(\hat{S}) \operatorname{dist}(Tx, \mathrm{N}(S) \cap \mathrm{R}(T))$$

for each $x \in \mathrm{D}(T)$, which proves that

$$\|S\| \gamma(\mathrm{R}(T), \mathrm{N}(S)) \geqq \gamma(\hat{S}).$$

This completes the proof of the theorem.

The following corollary is essentially proved by R. Bouldin in [1].

Corollary 6. *Let X, Y, Z be Banach spaces and let S be a bounded linear operator with* $\mathrm{D}(S) = Y$. *Assume that* $\mathrm{R}(S)$ *and* $\mathrm{R}(T)$ *are closed subspaces of Z and Y respectively. Then the following conditions are equivalent*:

(1) $\mathrm{R}(ST)$ *is a closed subspace of Z.*

(2) $\mathrm{N}(S) + \mathrm{R}(T)$ *is a closed subspace of Y.*

Proof. Since $\mathrm{R}(\hat{S}) = \mathrm{R}(ST)$, this follows from Corollary 3 and Theorem 5.

Corollary 7. *Under the same assumptions as in Corollary* 6, *we have the following estimate of $\gamma(\hat{S})$ in terms of the angular distance*:

$$\|S\| \alpha(\mathrm{N}(S)/Y_o, \mathrm{R}(T)/Y_o) \geqq \gamma(\hat{S})$$
$$\geqq (1/2)\gamma(S)\alpha(\mathrm{N}(S)/Y_o, \mathrm{R}(T)/Y_o)$$

where $Y_o = \mathrm{N}(S) \cap \mathrm{R}(T)$.

Proof. This follows from Theorems 2 and 5.

Finally, we note that the following estimate holds between $\gamma(\hat{S})$ and $\gamma(ST)$.

Theorem 8. $\gamma(ST) \geqq \gamma(\hat{S})\gamma(T)$.

Proof. For any $x \in \mathrm{D}(T)$, we have

$$\operatorname{dist}(Tx, \mathrm{N}(S) \cap \mathrm{R}(T)) \geqq \gamma(T) \operatorname{dist}(x, \mathrm{N}(ST))$$

by Lemma 1 in [3]. Hence we get

$$\|STx\| = \|\hat{S}(Tx)\| \geqq \gamma(\hat{S}) \operatorname{dist}(Tx, \mathrm{N}(\hat{S}))$$
$$= \gamma(\hat{S}) \operatorname{dist}(Tx, \mathrm{N}(S) \cap \mathrm{R}(T))$$
$$\geqq \gamma(\hat{S})\gamma(T) \operatorname{dist}(x, \mathrm{N}(ST))$$

for any $x \in \mathrm{D}(ST)$. This proves the desired estimate.

The following corollary, which is proved in [3], is immediate from Theorems 5 and 8.

Corollary 9. $\gamma(ST) \geqq \gamma(S)\gamma(T)\gamma(\mathrm{R}(T), \mathrm{N}(S))$.

References

[1] R. Bouldin: Closed range and relative regularity for products. J. Math. Anal. Appl., 61, 397–403 (1977).

[2] T. Kato: Perturbation Theory for Linear Operators. Die Grundlehen der math. Wissenschaften, Band 132, (2nd ed.), Springer-Verlag, Berlin and New York (1976).

[3] G. Nikaido: Remarks on the lower bound of a linear operator. Proc. Japan Acad., 56A, 321–323 (1980).