92. *Class Numbers of Positive Definite Binary and Ternary Unimodular Hermitian Forms*

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0. In this note we present some explicit formulae for the class numbers of positive definite unimodular hermitian forms of ranks two and three, over the rings of imaginary quadratic fields. The proofs, as well as some general discussion on the arithmetic of hermitian forms, will be given in a forthcoming paper [2].

1. Let K be an imaginary quadratic field, and R be its ring of integers. We denote by $d=d(K)$ the discriminant of K, and by t the number of distinct prime divisors of d. According to a result of Landherr [6], there exist 2^{t-1} mutually nonisometric classes of positive definite hermitian spaces $(V, \boldsymbol{H})$ $(\boldsymbol{H}: V\times V\to K)$ of a given rank n over K, which contain a unimodular R-lattice. They are parametrized by the system $\varepsilon=(\varepsilon_p)_{p|d}$ of local norm residues:

$$\varepsilon := \{\varepsilon_p = (d(V), K/\boldsymbol{Q})_p\,;\, p \mid d(K)\}$$

of the discriminant $d(V)=d(V,\boldsymbol{H})$ of $(V,\boldsymbol{H})$, which are subject to the condition $\prod_p \varepsilon_p=1$. Notice that, for $c\in \boldsymbol{Q}^\times$ the local norm residue of c at p is equal to $(c, K/\boldsymbol{Q})_p=(c, d(K))_p$ (=Hilbert symbol). Moreover, from a result of Jacobowitz [5], it is known that the set of unimodular R-lattices in $(V,\boldsymbol{H})$ consists of at most two genera with respect to the unitary group $G=U(V,\boldsymbol{H})$. One, which always exists, is called the genus of *normal* unimodular lattices, and denoted by $\mathcal{L}_o=\mathcal{L}_o(\varepsilon)$. The other, which exists only in case n is even and $2\mid d(K)$, is called the genus of *subnormal* or *even* unimodular lattices, and denoted by $\mathcal{L}_e=\mathcal{L}_e(\varepsilon)$.

2. Let $\mathcal{L}_o^1(\varepsilon)$, $\mathcal{L}_e^1(\varepsilon)$ be any genus with respect to the special unitary group $G^1:=\mathrm{SU}(V,\boldsymbol{H})$, which is contained in $\mathcal{L}_o(\varepsilon)$, $\mathcal{L}_e(\varepsilon)$ respectively. It is an interesting problem to ask for possible relations between the class numbers of them, and also between those of $\mathcal{L}_o(\varepsilon)$, $\mathcal{L}_e(\varepsilon)$. Suppose first that $n=\dim V=2$.

Theorem 1 ($n=2$). *The class number H^1 of binary unimodular hermitian R-lattices in $(V,\boldsymbol{H})$, $V=V(\varepsilon)$, with respect to the special unitary group* $\mathrm{SU}(V,\boldsymbol{H})$, *depends only on the G-genus $\mathcal{L}_o(\varepsilon)$ or $\mathcal{L}_e(\varepsilon)$. And H^1 is given by*

$$H^1=T_1+T_2+T_3,$$
$$T_1=(A/12)\prod_p (p+(-1/p))$$
$$T_2=(B/4)\prod_p (1+\varepsilon_p)$$

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$$T_3=(C/3)\prod_p(1+\varepsilon_p(-1/p)(-3/p))$$

where the products are extended over the prime divisors p of $d(K)$ such that $p\neq 2$, and the constants A, B, C are given in the following table: $(d=d(K))$.

		ε_2	$d=\text{odd}$	$4\Vert d$	$d\equiv 8$ (mod 32)	$d\equiv -8$ (mod 22)
$\mathcal{L}_o^1(\varepsilon)$	A	± 1	1	3	6	6
	B	± 1	1	3	2	2
	C	± 1	1	0	0	0
$\mathcal{L}_e^1(\varepsilon)$	A	$-$ \ $+$	* \ *	1 \ *	1 \ 3	3 \ 1
	B	$-$ \ $+$	* \ *	1 \ *	1 \ 1	1 \ 1
	C	$-$ \ $+$	* \ *	1 \ *	2 \ 0	0 \ 2

Remark 1. In the above formula, T_1, T_2, and T_3 are the contributions of the elements in $G^1=\mathrm{SU}(V, \boldsymbol{H})$, to the general formula in [1, 2] whose characteristic polynomials are $(X\pm 1)^2$, (X^2+1), and $(X^2\pm X+1)$ respectively.

Remark 2. The class number of definite binary hermitian matrices with coefficients in R and determinant one, has been studied by Hayashida [3]. The explicit formula for such class number, given in p. 40–41 [3], agrees with our result, in case where $\varepsilon=(1,\cdots,1)$ $(:=\varepsilon_o$, say). We shall describe the relation briefly in the next paragraph.

3. Let $n=\dim V$ be arbitrary and $\mathcal{L}=\mathcal{L}(\varepsilon)$ be a G-genus of R-lattices in $(V, \boldsymbol{H})$, $V=V(\varepsilon)$. Let L be a member of $\mathcal{L}$, and for any $M\in\mathcal{L}$, put $[L:M]:=\{\det(g)\,;\, g\in\mathrm{End}_K(V),\, Lg\subseteq M\}R$. Then the R-ideal $[L:M]$ depends only on the G^1-genus $\mathcal{L}^1$ to which M belongs (L is fixed).

Lemma 1. *Suppose that L is a free R-lattice. Then the following conditions on the G^1-genus $\mathcal{L}^1$ are equivalent*:

(i) *$\mathcal{L}^1$ contains a free R-lattice.*

(ii) *Any member of $\mathcal{L}^1$ is a free R-lattice.*

(iii) *$[L:M]$ is a principal ideal for any $M\in\mathcal{L}^1$.*

Moreover, if L is unimodular then one has $\varepsilon=\varepsilon_o$.

This is an easy consequence of the fact that the class number of $\mathrm{SL}_n(K)$ is one. Now let $\mathcal{H}_o^1$ (resp. $\mathcal{H}_e^1$) be the set of integral positive definite hermitian matrices $H\in M_n(R)$ such that $\det(H)=1$ and that the G.C.D. of their diagonal entries are 1 (resp. 2). Two members H_1, H_2 of them are called equivalent (resp. properly equivalent), if there exists $A\in\mathrm{GL}_n(R)$ (resp. $A\in\mathrm{SL}_n(R)$) such that $H_2=AH_1{}^t\bar{A}$.

Let $\mathcal{L}^f=\mathcal{L}_o^f$ (resp. $\mathcal{L}_e^f$) be the set of free R-lattices in $\mathcal{L}_o(\varepsilon_o)$ (resp. $\mathcal{L}_e(\varepsilon_o)$), the genus of unimodular R-lattices. For any member M of $\mathcal{L}^f$, we take an R-basis $(x_i)_{i=1}^n$ of M and put $H=(\boldsymbol{H}(x_i, x_j))$. Then we see that H belongs to $\mathcal{H}_o^1$ (resp. $\mathcal{H}_e^1$), and that the equivalence class of H does not depend on the choice of basis. Moreover, one has

Lemma 2. *The above correspondence $M \mapsto H$ induces the canonical bijections $\mathcal{L}_o^f/G \cong \mathcal{H}_o^1/$(equivalence), $\mathcal{L}_e^f/G \cong \mathcal{H}_e^1/$(equivalence). Here $\mathcal{L}^f/G$ denotes the set of G-classes, or G-orbits, in $\mathcal{L}^f$.*

Lemma 3. *Let $\mathcal{L}^R$ be the set of R-lattices M in $\mathcal{L}^f$ such that $[L:M]=R$, where L is a fixed member of $\mathcal{L}^f$. Also put $G^u := \{g \in G ; \det(g) \in R^\times\}$. Then the inclusion map induces a bijection: $\mathcal{L}^R/G^u \cong \mathcal{L}^f/G$.*

Lemma 4. *The set $\mathcal{L}^R$ consists of a single G^1-genus, except for the case where $4 \| d(K)$, and $n \equiv 0 \pmod 4$, $\mathcal{L}^R = \mathcal{L}_e^R$. In the exceptional case $\mathcal{L}_e^R$ consists of two G^1-genera.*

Proposition 1. *There exist bijections $\mathcal{L}_o^R/G^1 \xrightarrow{\cong} \mathcal{H}_o^1/$(proper equiv.), and $\mathcal{L}_e^R/G^1 \xrightarrow{\cong} \mathcal{H}_e^1/$(proper equiv.), for which the following diagram is commutative ($\mathcal{H}^1 = \mathcal{H}_o^1$ or $\mathcal{H}_e^1$):*

$$\begin{array}{ccc} \mathcal{L}^R/G^1 & \xrightarrow{\quad\cong\quad} & \mathcal{H}^1/(\text{proper equiv.}) \\ \downarrow & & \downarrow \\ \mathcal{L}^R/G^u \cong \mathcal{L}^f/G & \xrightarrow{\cong} & \mathcal{H}^1/(\text{equivalence}) \end{array}$$

A simple argument using these results now gives:

Theorem 2. *Let $\mathcal{L}$ be a G-genus of unimodular R-lattices in $(V, \boldsymbol{H})$, and suppose that, in the above notations,* (i) $(n, h(K)/2^{t-1})=1$ [$h(K):=$ *class number of K*], (ii) *$\mathcal{L}^R$ consists of a single G^1-genus* (*see Lemma* 4), *and* (iii) *for any $M \in \mathcal{L}$ and $\zeta \in R^\times$, there exists $\gamma \in G$ such that* $\det(\gamma)=\zeta$. *Then one has, for any G^1-genus $\mathcal{L}^1$ contained in $\mathcal{L}$, $\sharp[\mathcal{L}/G] = \sharp[\mathcal{L}^1/G^1]\cdot(h(K)/2^{t-1})$.*

Remark 3. If n is odd and $K \neq \boldsymbol{Q}(\sqrt{-3})$, then the conditions (ii) and (iii) are automatically satisfied.

4. Now we suppose that $n=3$ and $\varepsilon=\varepsilon_0$, so that $(V, \boldsymbol{H})$ is isometric to the standard hermitian space $\boldsymbol{H}(x,y)=\sum_{i=1}^3 x_i\bar{y}_i$, and $\mathcal{L}$ (=the principal genus) is represented by the standard lattice R^3. Our main results in this case are:

Theorem 3. *The class number H^1 of the principal genus $\mathcal{L}^1$ ($\ni R^3$) with respect to* SU$(V, \boldsymbol{H})$ *is given as follows: H^1=add (resp. even) if $d(K)=-3, -4$ (resp. $-7, -8$); and otherwise*

$$H^1 = T_1 + T_2 + T_3 + T_{41} + T_{42},$$

where

$$\begin{aligned} T_1 &= B_{3,\chi}/144 \\ T_2 &= (h(K)/48)[4|d(K)|-1-3\chi_K(2)], \\ T_3 &= (h(K)/8)[3+\chi_K(2)+\{1+(2,K/\boldsymbol{Q})_2\}\{1+(5,K/\boldsymbol{Q})_2\}], \\ T_{41} &= (h(K)/12)[7-\chi_K(3)] \\ T_{42} &= (h(K)/12)[1+\chi_K(3)], \end{aligned}$$

where $\chi(n)=\chi_K(n)=(d(K)/n)$ denotes the Dirichlet character attached to K, and $B_{m,\chi}$ is the m-th generalized Bernoulli number associated with χ_K.

Remark 4. Again, T_i represent the contributions from the characteristic polynomials

$$
\begin{aligned}
f_1(X)&=(X-1)^3,\\
f_2(X)&=(X-1)(X+1)^2,\\
f_3(X)&=(X-1)(X^2+1),\\
f_{41}(X)&=(X-1)(X^2+X+1),\\
f_{42}(X)&=(X-1)(X^2-X+1)
\end{aligned}
$$

to the general formula (cf. [1], [2]). The same method applied to the group $G=U(V,\boldsymbol{H})$ gives by comparison the following:

Theorem 4. *The class numbers H, H^1 of the ternary principal genera with respect to $U(V,\boldsymbol{H})$ and* SU $(V,\boldsymbol{H})$ *are related as*

$$H=(h(K)/2^{t-1})\cdot H^1.$$

5. Examples (class numbers of the principal genera).

$d(K)$	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24	-31	-35	-39	-40	-43
$H^1(n=2)$	1	1	1	1	2	2	2	3	3	2	3	4	4	4	4
$H^1(n=3)$	1	1	2	2	2	5	3	7	10	7	13	10	21	12	8

$d(K)$	-47	-51	-52	-55	-56	-59	-67	-68	-71	-79	-83	......	-163
$H^1(n=2)$	5	4	5	6	4	6	6	6	7	7	8	......	14
$H^1(n=3)$	31	16	18	31	31	26	17	42	66	59	43	......	111

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