91. Class Number Relations of Algebraic Tori. II

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Let k be an algebraic number field and K be its finite extension. Recently, T. Ono [5] defined a positive rational number $E(K/k)$ and obtained an equality between $E(K/k)$ and some cohomological invariants, when K is a normal extension of k. He also defined a rational number $E'(K/k)$ and we obtained in our paper [2] a similar equality between $E'(K/k)$ and some cohomological invariants. Here we shall generalize these equalities for any non-normal extensions. This note is a continuation of our preceding paper [2], to which we refer the reader for terminology and notations. In the following, all the cohomology groups $H^r(K/k, A_K)$ means the non-normal cohomology groups of I. T. Adamson $H^r(A_K|A_k)$.

§1. First, consider the following exact sequence of algebraic tori defined over k

$$(1)\qquad 0\longrightarrow R^{(1)}_{K/k}(G_m)\stackrel{a}{\longrightarrow}R_{K/k}(G_m)\stackrel{N}{\longrightarrow}G_m\longrightarrow 0,$$

where N is the norm map for K/k. For the sake of simplicity, we shall abbreviate $R_{K/k}(G_m)$, $R^{(1)}_{K/k}(G_m)$, G_m to T, T', T'' in this section. Let L be a normal extension of k containing K. Then L is a splitting field of T, T', T''. We denote Gal(L, k) by G and Gal(L/K) by H. We define a k-morphism $b: T\rightarrow T'$ by putting

$$b(x)=x^m(Nx)^{-1},\qquad \text{where } m=[K:k].$$

Then $c=b\times N: T\rightarrow T'\times T''$ and $m=b\cdot a: T'\rightarrow T'$ are k-isogenies, where m is the map $m(x)=x^m$. From the definition of $E(K/k)$ and Theorem of [2], we have

$$(2)\qquad E(K/k)=\frac{\tau(T)\times q(\hat{m}(k))}{\tau(T')\tau(T'')q(\hat{c}(k))}\times\frac{\prod_p[\mathrm{Ker}\,(H^1(G_{P(L)}, T'(O_{P(L)}))\rightarrow H^1(G_{P(L)}, T(O_{P(L)})))]}{[\mathrm{Ker}\,(H^1(G, T'(O_L))\rightarrow H^1(G, T(O_L)))]},$$

where p runs over all the places of k and $P(L)$ is an extension of p to L and $G_{P(L)}$ is the decomposition group of $P(L)$. It is known that $\tau(T)=\tau(T'')=1$ and

$$\tau(T')=[K_0:k]/[\mathrm{Ker}\,(H^0(K/k, K^\times)\rightarrow H^0(K/k, K^\times_A))]\qquad \text{in (2)},$$

where K_0 is the maximal abelian extension of k contained in K. From the fact that T' is an anisotropic torus, we have $q(\hat{m}(k))=1$ and $q(\hat{c}(k))=1$. On the other hand, we have

$$\mathrm{Ker}\,(H^1(G, T'(O_L))\rightarrow H^1(G, T(O_L)))$$
$$\cong\mathrm{Cok}\,(T(O_L)^G\rightarrow T''(O_L)^G)$$

$$\cong \mathrm{Cok}\,(O_K^\times \to O_k^\times)$$
$$\cong O_k^\times / N_{K/k} O_K^\times$$
$$\cong H^0(K/k, O_K^\times).$$

In the same way as above, we have

$$\mathrm{Ker}\,(H^1(G_{P(L)}, T'(O_{P(L)})) \to H^1(G_{P(L)}, T(O_{P(L)})))$$
$$\simeq O_p^\times / N_{K_P/k_p} O_P^\times$$
$$\cong H^0(K_P/k_p, O_P^\times),$$

where P is an extension of p to K. From (2), we have the following

Theorem 1. *For any finite extension K/k, the Euler number $E(K/k)$ is written in the form*

$$E(K/k) = \frac{[\mathrm{Ker}\,(H^0(K/k, K^\times) \to H^0(K/k, K_A^\times))] \prod_p [H^0(K_P/k_p, O_P^\times)]}{[K_0 : k][H^0(K/k, O_K^\times)]}.$$

Remark 1. We note here that above equation is formally obtained from replacing cohomology groups in T. Ono's theorem of [5] by corresponding non-normal cohomology groups. Without using cohomology groups, the above formula is written as follows

$$E(K/k) = \frac{[k^\times \cap N_{K/k} K_A^\times : N_{K/k} K^\times] \prod_p e_p^0}{[K_0 : k][O_k^\times : N_{K/k} O_K^\times]},$$

where e_p^0 denotes the ramification index of the maximal abelian extension over k_p which is contained in K_P.

§ 2. Now, consider the following exact sequence of algebraic tori

$$(3) \qquad 0 \longrightarrow G_m \overset{d}{\longrightarrow} R_{K/k}(G_m) \overset{f}{\longrightarrow} R_{K/k}(G_m)/G_m \longrightarrow 0,$$

where $f(x) = x \bmod G_m (x \in R_{K/k}(G_m))$. In this section, we shall abbreviate $R_{K/k}(G_m), G_m, R_{K/k}(G_m)/G_m$ to T, T', T''. In the same way as in § 1, there exist k-isogenies

$$g = N \times f : T \longrightarrow T' \times T'' \quad \text{and} \quad m = N \cdot d : T' \longrightarrow T'.$$

Here m is the map $m(x) = x^m$ ($x \in T' = G_m$). Then, from Theorem of [2], the number $E'(K/k)$ is written in the form

$$(4) \qquad E'(K/k) = \frac{\tau(T) \times q(\hat{m}(k))}{\tau(T')\tau(T'')q(\hat{g}(k))} \times \frac{\prod_p [\mathrm{Ker}\,(H^1(G_{P(L)}, T'(O_{P(L)})) \to H^1(G_{P(L)}, T(O_{P(L)})))]}{[\mathrm{Ker}\,(H^1(G, T'(O_L)) \to H^1(G, T(O_L)))]},$$

where $\tau(T) = \tau(T') = 1$, $\tau(T'') = m$, $q(\hat{g}(k)) = 1$ and $q(\hat{m}(k)) = m$. On the other hand, we have

$$\mathrm{Ker}\,(H^1(G, T'(O_L)) \to H^1(G, T(O_L)))$$
$$= \mathrm{Ker}\,(H^1(G, O_L^\times) \to H^1(H, O_L^\times))$$
$$\cong H^1([G : H], O_L^\times) = H^1(K/k, O_K^\times).$$

(See [1], Theorem 7.3.) In the same way as above, we have

$$\mathrm{Ker}\,(H^1(G_{P(L)}, T'(O_{P(L)})) \to H^1(G_{P(L)}, T(O_{P(L)})))$$
$$= \mathrm{Ker}\,(H^1(G_{P(L)}, O_{P(L)}^\times) \to H^1(G_{P(L)} \cap H, O_{P(L)}^\times))$$
$$\cong H^1(K_P/k_p, O_P^\times).$$

Combining these, we have

Theorem 2. *For any finite extension K/k, we have*

$$E'(K/k)=\frac{[H^1(K/k, U_K)]}{[H^1(K/k, O_K^\times)]}.$$

Remark 2. $[H^1(K/k, U_K)]=\prod_p [H^1(K_P/k_p, O_P^\times)]=\prod_p e_p$, where e_p is the ramification index of P.

Remark 3. In the next paper, we shall show the relations between $E(K/k)$, $E'(K/k)$ and other invariants for K/k (the central class number, the ambiguous ideal class number etc.).

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