90. A Remark on the λ-invariant of Real Quadratic Fields

By Takashi FUKUDA,*) Keiichi KOMATSU,**) and Hideo WADA***)

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 13, 1986)

In previous papers [1] and [2] by two of us, we considered Greenberg's conjecture (cf. [3]) on real quadratic case. In [2], it was essential to assume $n_1<n_2$ for two natural numbers n_1 and n_2 whose definitions will be recalled in the following. The purpose of this paper is to give some examples concerning the case $n_1=n_2=2$.

Let k be a real quadratic field with class number h_k and p an odd prime number which splits in $k/\boldsymbol{Q}$. Let $\mathfrak{p}$ be a prime factor of p in k, and ε be a fundamental unit of k. Choose $\alpha\in k$ such that $\mathfrak{p}^{h_k}=(\alpha)$. We define n_1 (resp. n_2) to be the maximal integer such that $\alpha^{p-1}\equiv 1 \pmod{p^{n_1}\boldsymbol{Z}_p}$ (resp. $\varepsilon^{p-1}\equiv 1 \pmod{p^{n_2}\boldsymbol{Z}_p}$). Note that n_1 is uniquely determined under the condition $n_1\leqq n_2$. For the cyclotomic $\boldsymbol{Z}_p$-extension

$$k=k_0\subset k_1\subset k_2\subset\cdots\subset k_n\subset\cdots\subset k_\infty,$$

let A_n be the p-primary part of the ideal class group of k_n, B_n the subgroup of A_n consisting of ideal classes which are invariant under the action of $\mathrm{Gal}(k_n/k)$, and D_n the subgroup of A_n consisting of ideal classes which contain a product of ideal lying over p. Let E_n be the unit group of k_n. For $m\geqq n\geqq 0$, $N_{m,n}$ denotes the norm maps from k_m to k_n, we shall give a proof for the sake of completeness.

Lemma. *Let k be a real quadratic field and p an odd prime number which splits in $k/\boldsymbol{Q}$. Assume that*

(1) $n_1=n_2=2$,

(2) $|A_0|=1$, *and*

(3) $N_{1,0}(E_1)=E_0$.

Then we have $|A_n|=|A_1|$ for all $n\geqq 1$ and in particular $\mu_p(k)=\lambda_p(k)=0$, where μ, λ denote the Iwasawa invariants.

Proof. From Proposition 1 of [1], $|B_n|=p$ for all $n\geqq 1$. By the assumptions (2) and (3), we have

$$|D_1|=\frac{p}{(E_0\,;\,N_{1,0}(E_1))}=p.$$

It follows that $B_n=D_n$ and $N_{n+1,n}:B_{n+1}\to B_n$ are isomorphisms for all $n\geqq 1$. Now, $N_{n+1,n}:A_{n+1}\to A_n$ is surjective and its restriction to B_{n+1} is injective. Hence, $N_{n+1,n}:A_{n+1}\to A_n$ are isomorphisms for all $n\geqq 1$.

When $p=3$, k_1 is a real cyclic extension of degree 6 over $\boldsymbol{Q}$. In this case, we can determine a system of fundamental units of k_1 for a given k

*) Department of Mathematics, Faculty of Science, Yamagata University.
**) Department of Mathematics, Tokyo University of Agriculture and Technology.
***) Department of Mathematics, Sophia University.

by the method of Mäki [4] and hence determine whether $N_{1,0}(E_1)$ is E_0 or not. We obtain the following Theorem by excuting Mäki's algorithm for those m's which are in the table of [2] and satisfy $n_1=n_2=2$.

Theorem. *Let* $p=3$ *and* $k=\boldsymbol{Q}(\sqrt{m})$ *where* $m=103, 139, 418, 679, 727, 790, 1153, 1261, 1609, 1642$, or 1726. *Then the assumptions* (1), (2) *and* (3) *of lemma are satisfied for these* k*'s. Hence* $\mu_3(k)=\lambda_3(k)=0$ *for the above values of* m*'s.*

Remark. Let $k^*=k(e^{2\pi i/p})$. We remark that $N_{1,0}(E_1)=E_0^p$ if $\lambda_p^-(k^*)=1$. For k's in our Theorem, we have $\lambda_3^-=4$ for $m=1609$ and $\lambda_3^-=2$ for others.

References

[1] T. Fukuda and K. Komatsu: On the λ invariants of $\boldsymbol{Z}_p$-extensions of real quadratic field. J. Number Theory, **23**, 238–242 (1986).

[2] ——: On $\boldsymbol{Z}_p$-extensions of real quadratic fields. J. Math. Soc. Japan, **38**, 95–102 (1986).

[3] R. Greenberg: On the Iwasawa invariants of totally real number fields. Amer. J. Math., **98**, 263–284 (1976).

[4] S. Mäki: The determination of units in real sextic fields. Springer Lecture Notes in Math., vol. 797 (1980).