89. *String Theories and Prime Number Theories*

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 13, 1986)

String theories (cf. Green-Gross [5]) seem to suggest much to prime number theories. We supplement the previous report [7] from the view point of string theories. In § 1 we notice an application of string theories, and we remark the "multiple string case" in § 2. The contents of this report were presented in a symposium entitled "Superstring Theories" at the University of Tokyo in September 1986. Some details will appear elsewhere.

§ 1. An application of string theories. Let M be a compact Riemann surface of genus 2, τ_M the period matrix belonging to the Siegel upper half space of genus 2, and $Z_M(s)$ the original Selberg zeta function. Let χ_{10} be the Siegel cusp form of genus 2 and weight 10 uniquely determined up to constant: $\chi_{10}=c\prod_{m:\mathrm{even}}\vartheta_m^2$. Then:

Theorem 1. *$Z'_M(1)^{13}Z_M(2)^{-1}=C\,|\chi_{10}(\tau_M)|^2(\det \operatorname{Im}\tau_M)^{10}$ up to an absolute constant C independent of M.*

This follows from the recent progress of string theories. First, D'Hoker-Phong [4] (cf. [1]) showed that:

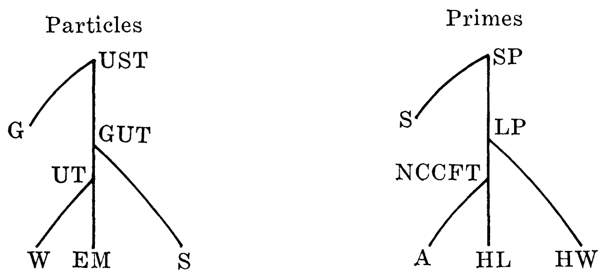
$$Z'_M(1)^{13}Z_M(2)^{-1}=C_1(\det \Delta_M)^{13}(\det \Delta_M^{1,+})^{-1},$$

where Δ_M is the usual Laplacian on M and $\Delta_M^{1,+}$ is a Laplacian acting on certain tensors. Secondly, Belavin-Knizhnik [3] (cf. [2], [6], [10]) showed that:

$$|\chi_{10}(\tau_M)|^2(\det \operatorname{Im}\tau_M)^{10}=C_2(\det \Delta_M)^{13}(\det \Delta_M^{1,+})^{-1}.$$

These two results are proved by quite different methods, and Theorem 1 seems to be astonishing.

We notice another suggestion from string theories, which is conjecturally schematized as follows:



The left tree indicates the unification of the four forces: weak, electromagnetic, strong, and gravitational. The right tree indicates the unification of the four zeta functions: Artin, Hecke-Langlands, Hasse-Weil, and

Selberg. We omit the detailed comparison except for emphasizing two points: (1) Selberg zeta functions treat closed strings (real one dimensional loops) as "primes" in contrast with other three zeta functions where "primes" are zero dimensional closed points, and (2) gauge groups correspond to (extended) Galois groups. For a background we refer to [9].

§ 2. A multiple zeta function. Let $X:\boldsymbol{R}^r\to\mathrm{Aut}(M)$ be a group homomorphism given by $t\mapsto X_t$ from the additive group of r-copies of real numbers to the automorphism group of a set M. Such an X is called an abstract dynamical system. (It is usual to suppose suitable topological or analytical conditions on X.) We define the zeta function $\zeta(s,X)$ of X as follows. We say that an orbit $p=\boldsymbol{R}^r\cdot m$ for an $m\in M$ is "periodic" if the isotropy subgroup (stabilizer) $\boldsymbol{R}^r_p=\boldsymbol{R}^r_m$ is isomorphic to $\boldsymbol{Z}^r$, where $\boldsymbol{Z}$ denotes the additive group of integers. We denote by $\mathrm{Per}(X)$ the set of all periodic orbits, and for each $p\in\mathrm{Per}(X)$ we define $N(p)=\exp(\mathrm{vol}(\boldsymbol{R}^r/\boldsymbol{R}^r_p))$ where vol denotes the volume. Then we put

$$\zeta(s,X)=\prod_{p\in\mathrm{Per}(X)}(1-N(p)^{-s})^{-1}$$

where s is a variable complex number. This definition was noted in [7, Remark 1]. When $r=1$, this zeta function coincides with the zeta function of Selberg-Smale-Ruelle, and there are vast results in this case. On the contrary, it seems that there is no paper treating zeta functions for $r\geqq 2$. We investigate the latter case in a particularly simple "completely reducible" situation. (On the level of zeta functions of analytic rings of [7], "reducible" means nothing but "decomposable into a tensor product".)

We say that an abstract dynamical system $X:\boldsymbol{R}^r\to\mathrm{Aut}(M)$ is completely reducible if there are dynamical systems $X^i:\boldsymbol{R}\to\mathrm{Aut}(M_i)$ for $i=1,\cdots,r$ such that $M=M_1\times\cdots\times M_r$ and $X_t(m_1,\cdots,m_r)=(X^1_{t_1}(m_1),\cdots,X^r_{t_r}(m_r))$ for $(m_1,\cdots,m_r)\in M$ and $t=(t_1,\cdots,t_r)\in\boldsymbol{R}^r$. We call X the product of $X^1,\cdots,X^r$. In this case $\mathrm{Per}(X)$ is identified with $\mathrm{Per}(X^1)\times\cdots\times\mathrm{Per}(X^r)$ ("multiple prime strings") and we have $N(p)=\exp(\log N(p_1)\cdots\log N(p_r))$ for $p=(p_1,\cdots,p_r)\in\mathrm{Per}(X)$. Thus the study of $\zeta(s,X)$ is reformulated as follows. Let $P_1,\cdots,P_r$ be prime sets in the sense of [8] with norm functions $N_i:P_i\to\boldsymbol{R}$. Define $N:P_1\times\cdots\times P_r\to\boldsymbol{R}$ by $N(p_1,\cdots,p_r)=\exp(\log N_1(p_1)\cdots\log N_r(p_r))$ for $p_i\in P_i$. Then $P_1\times\cdots\times P_r$ is a prime set again, and our object is the study of the multiple Euler product

$$\zeta(s,P_1\times\cdots\times P_r)=\prod_{(p_1,\cdots,p_r)}(1-N(p_1,\cdots,p_r)^{-s})^{-1},$$

which may be called a multiple zeta function. (We notice that there exist various "multiple zeta functions" also.) We make a trivial remark that every prime set P is obtainable from an abstract dynamical system $X:\boldsymbol{R}\to\mathrm{Aut}(M)$ via $P=\mathrm{Per}(X)$ so $\zeta(s,P)=\zeta(s,X)$: for example, it is sufficient to take $M=P\times(\boldsymbol{R}/\boldsymbol{Z})$ and put

$$X_t(p,x)=(p,x+t/\log N(p)\bmod 1).$$

For simplicity we note the following two cases.

Theorem 2. *Let M_1 and M_2 be compact Riemann surfaces of genus greater than* 1. *Let $X^i : \boldsymbol{R} \to \mathrm{Aut}(M_i)$ be the geodesic action, and $X : \boldsymbol{R}^2 \to \mathrm{Aut}(M_1 \times M_2)$ be the product dynamical system. Then $\zeta(s, X) = \zeta(s, \mathrm{Per}(X^1) \times \mathrm{Per}(X^2))$ is meromorphic in $\mathrm{Re}(s) > 0$ with the natural boundary $\mathrm{Re}(s) = 0$.*

Theorem 3. *Let $P(\boldsymbol{Z})$ be the prime set of rational primes. Then $\zeta(s, P(\boldsymbol{Z}) \times P(\boldsymbol{Z}))$ is meromorphic in $\mathrm{Re}(s) > 0$ with the natural boundary $\mathrm{Re}(s) = 0$.*

The method of the proof of Theorems 2 and 3 is a simple modification of [8].

Looking the "largest" pole of $\zeta(s, X)$, we have an asymptotic distribution of multiple primes $(p_1, \cdots, p_r)$. For example $\zeta(s, P(\boldsymbol{Z}) \times P(\boldsymbol{Z}))$ is non-zero holomorphic in $\mathrm{Re}(s) \geqq 1/\log 2$ except for the double pole at $s = 1/\log 2$, so we have

$$\sharp\{(p_1, p_2) ; p_i \in P(\boldsymbol{Z}), N(p_1, p_2) \leqq t\} \sim 2 \log 2 \frac{t^{1/\log 2}}{\log t}$$

as $t \to \infty$. Similarly, let M be a compact Riemann surface of genus greater than 1 with the geodesic action $X : \boldsymbol{R} \to \mathrm{Aut}(M)$, then $\zeta(s, \mathrm{Per}(X) \times \mathrm{Per}(X))$ is non-zero holomorphic in $\mathrm{Re}(s) \geqq 1/l(M)$ except for the double pole at $s = 1/l(M)$,

$$l(M) = \min\{\log N(p) ; p \in \mathrm{Per}(X)\}$$

being the minimal length of a closed geodesic on M, and we have

$$\sharp\{(p_1, p_2) ; p_i \in \mathrm{Per}(X), N(p_1, p_2) \leqq t\} \sim 2 \cdot l(M) \frac{t^{1/l(M)}}{\log t} \qquad \text{as } t \to \infty.$$

Remark. For various applications analytic dynamical systems $X : \boldsymbol{R}^r \to \mathrm{Aut}(M)$ are important. We obtain a natural analytic dynamical system of the above form for $M = \Gamma \backslash G / K$ where G is a Lie group of rank r, K is a maximal compact subgroup of G, and Γ is a co-compact discrete subgroup of G. For example, the situation of Theorem 2 is obtained from $G = \mathrm{SL}(2, \boldsymbol{R}) \times \mathrm{SL}(2, \boldsymbol{R})$. More generally we must study quantum dynamical systems as in [7].

References

[1] M. A. Baranov and A. S. Svarc: Multiloop contribution to string theory. JETP Lett., **42**, 419–421 (1985).
[2] A. Belavin, V. Knizhnik, A. Morozov, and A. Perelomov: Two and three loop amplitudes in bosonic string theory. ibid., **43**, 319–321 (1986).
[3] A. Belavin and V. Knizhnik: Complex geometry and theory of quantum strings. Landau Institute (1986) (preprint).
[4] E. D'Hoker and D. H. Phong: Multiloop amplitudes for the bosonic Polyakov string. Nucl. Phys., B **269**, 205–234 (1986).
[5] M. Green and D. Gross (eds.): Unified String Theories. World Scientific, Singapore (1986).
[6] A. Kato, Y. Matsuo, and S. Odake: Modular invariance and two-loop bosonic string vacuum amplitude. University of Tokyo (1986) (preprint).

[7] N. Kurokawa: Zeta functions of analytic rings via Euler products. Proc. Japan Acad., **62A**, 193–196 (1986).
[8] ——: On the meromorphy of Euler products (I); (II). Proc. London Math. Soc., **53**, 1–47; 209–236 (1986).
[9] ——: Elementary particles and prime numbers. Surikagaku, No. 281 (1986) (in Japanese).
[10] G. Moore: Modular forms and two-loop string physics. Phys. Lett., **176**, 369–379 (1986).