86. Products of Compact Fréchet Spaces

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 13, 1986)

1. Introduction. A topological space X is called *Fréchet* if each point in the closure of a subset $A\subset X$ is the limit of a sequence from A. In 1972, E. Michael [5] raised the question whether there exist two compact Hausdorff spaces X and Y such that the product space $X\times Y$ is not Fréchet. After then, various examples of such spaces were constructed by T. K. Boehme and M. Rosenfeld [1] (under the continuum hypothesis CH), V. I. Malyhin and B. E. Šapirovskiĭ [4], R. C. Olson [7] (under Martin's axiom), and P. Simon [8] (without extra set-theoretic assumptions).

Generalizing E. Michael's question for more than two compact spaces, T. Nogura [6] asked: For $n\geqq 2$, is there a compact Fréchet space X such that X^n is Fréchet but X^{n+1} is not Fréchet?

The purpose of this paper is to answer the question positively under Martin's axiom. Indeed, using G. Gruenhage's technique [2] and Franklin compact spaces, we construct for each n with $3\leqq n\leqq\omega$, a compact Fréchet space X such that X^k is Fréchet for any $k<n$, but X^n is not Fréchet.

All spaces are assumed to be Hausdorff. The symbol ω denotes the first infinite cardinal and, simultaneously, the set of non-negative integers with the discrete topology.

2. Preliminaries. The Stone-Čech compactification of the countable discrete space ω is denoted by $\beta\omega$. For each $A\subset\omega$, the set A^* is defined by $A^*=\mathrm{cl}_{\beta\omega}A-A$. Let $\mathcal{P}$ be an infinite family of disjoint clopen subsets of ω^*. The *Franklin compact space* $F(\mathcal{P})$ is a quotient space of $\beta\omega$ obtained by the decomposition of $\beta\omega$ into $\{\omega^*-\bigcup\mathcal{P}\}$, elements of $\mathcal{P}$, and one-point sets $\{n\}$ with $n\in\omega$. Express $F(\mathcal{P})$ as $\{\infty\}\cup\mathcal{P}\cup\omega$, or more precisely, as $\{\infty_{\mathcal{P}}\}\cup\mathcal{P}\cup\omega$.

Note that every family $\mathcal{P}$ of disjoint clopen subsets of ω^* can be written as $\mathcal{P}=\{I^*: I\in\mathcal{J}\}$, where the family $\mathcal{J}$ is an almost disjoint family of infinite subsets of ω. A family $\mathcal{J}$ is said to be *almost disjoint* if $I\cap J$ is finite for any distinct members $I, J\in\mathcal{J}$.

It is easy to check the following lemma:

Lemma 1. (a) *$F(\mathcal{P})$ is a compact Hausdorff space.*

(b) *Each point of ω is isolated.*

(c) *Let $P\in\mathcal{P}$ and $P=I^*$, where $I\in\mathcal{J}$. Then the family $\{\{P\}\cup(I-F): F$ is a finite subset of $\omega\}$ is a neighborhood base at the point $\{P\}$.*

(d) *The family $\{\{\infty_{\mathcal{P}}\}\cup(\mathcal{P}-Q)\cup(\omega-\bigcup Q-F): Q$ is a finite subfamily of $\mathcal{P}$, and F is a finite subset of $\omega\}$ is a neighborhood base at the point $\infty_{\mathcal{P}}$.*

The following lemma was essentially proved by T. K. Boehme and M. Rosenfeld [1], and V. I. Malyhin and B. E. Šapirovskiĭ [4]. G. Gruenhage [2] proved the case $n=\omega$.

Lemma 2. *Let $1\leqq n\leqq\omega$. Suppose that $\mathcal{P}=\bigcup\{\mathcal{P}_i : i<n\}$ is an infinite maximal disjoint family of clopen subsets of ω^* such that $\mathcal{P}_i\cap\mathcal{P}_j=\emptyset$ whenever $i\neq j$. Then the product space $\prod\{\{\infty_{F(\mathcal{P}_i)}\}\cup\omega : i<n\}$ is not a Fréchet space as a subspace of the product $\prod\{F(\mathcal{P}_i) : i<n\}$ of Franklin compact spaces.*

A space X has *countable tightness* if for each $A\subset X$ and $x\in \mathrm{cl}\,A$, there is a countable subset B of A such that $x\in\mathrm{cl}\,B$.

Lemma 3 (V. I. Malyhin [3]). *Suppose that X_i is a compact space with countable tightness for each $i\in\omega$. Then the product $\prod\{X_i : i\in\omega\}$ also has countable tightness.*

G. Gruenhage constructed the following maximal almost disjoint family in order to show the existence of a countable Fréchet space X such that X^n is Fréchet for all $n\in\omega$, but X^ω is not Fréchet.

Lemma 4 [2] (MA). *There is an infinite maximal almost disjoint family $\mathcal{I}=\bigcup\{\mathcal{I}_i : i\in\omega\}$ of infinite subsets of ω satisfying the following properties:*

(a) *$\mathcal{I}_i\cap\mathcal{I}_j=\emptyset$ if $i\neq j$;*

(b) *each finite subset of ω is contained in a member of $\mathcal{I}_i$ for each $i\in\omega$; and*

(c) *suppose that $k\in\omega$, $J\subset k$, $\{I_j : j\in J\}\subset\mathcal{I}$ and $A\subset(\prod\{\omega : i\in k-J\})\times(\prod\{I_j : j\in J\})$ are such that*

$$A\cap[(\prod\{\omega-E(i) : i\in k-J\})\times(\prod\{I_j-F(j) : j\in J\})]\neq\emptyset$$

whenever $E(i)$ is a finite union of members of $\mathcal{I}$, and $F(j)$ is a finite subset of ω. Then for each $m\in\omega$, there is a sequence $\{a_0, a_1, \cdots\}$ in A such that whenever $E(i)$ is a finite union of members of $\mathcal{I}-\mathcal{I}_m$ and $F(j)$ is a finite subset of ω, there is a natural number n satisfying that

$$\{a_n, a_{n+1}, a_{n+2}, \cdots\}\subset(\prod\{\omega-E(i) : i\in k-J\})\times(\prod\{I_j-F(j) : j\in J\}).$$

Proof (Sketch). We shall follow the construction of $\mathcal{I}=\bigcup\{\mathcal{I}_i : i\in\omega\}=\{I_i(\alpha) : \alpha<\mathfrak{c},\ i\in\omega\}$ and the notation in [2]. Obviously the property (a) holds. By the construction, for each $i\in\omega$ and a finite subset F of ω, there is $n\in\omega$ with $F\subset I_i(n)\in\mathcal{I}_i$. Therefore (b) is satisfied. It remains to check (c). Suppose the assumption of (c). Similarly to the last paragraph of the proof of Theorem in [2], take $\kappa<\mathfrak{c}$, $\beta_0=\beta(\kappa)$, the sequence $\vec{x}_0, \vec{x}_1, \cdots$ in $A=A_{\beta(\kappa)}$ and the partition $\omega=\bigcup\{W_m : m\in\omega\}$ at the κ-stage. Define $\{a_0, a_1, \cdots\}=\{\vec{x}_n : n\in W_m\}$. Observe that $I_m(\kappa)=\{\pi_i(\vec{x}_n) : n\in W_m,\ i\in k-J\}\in\mathcal{I}_m$, and hence for each $I\in\mathcal{I}-\mathcal{I}_m$, $I\cap I_m(\kappa)$ is finite because $\mathcal{I}$ is an almost disjoint family. Now it is not difficult to check the property (c).

3. Example.

Lemma 5 (MA). *For each n with $3\leqq n\leqq\omega$, there is a family $\{X_i : i<n\}$ of compact Fréchet spaces such that*

(a) *if $k<n$ and $\{Y_i : i<k\}$ is a family where each Y_i is equal to X_j for*

some $j<n$, then the product $\prod\{Y_i : i<k\}$ is Fréchet, but

(b) *$\prod\{X_i : i<n\}$ is not Fréchet.*

Proof. Let $3\leqq n\leqq\omega$ and $\mathcal{J}=\bigcup\{\mathcal{J}_i : i\in\omega\}$ be the family in Lemma 4. If $n<\omega$, define $\mathcal{P}_i=\{I^* : I\in\mathcal{J}_i\}$ for each $i<n-1$ and $\mathcal{P}_{n-1}=\{I^* : I\in\bigcup\{\mathcal{J}_i : n-1\leqq i\in\omega\}\}$. If $n=\omega$, simply define $\mathcal{P}_i=\{I^* : I\in\mathcal{J}_i\}$ for each $i<n=\omega$. Define $X_i=F(\mathcal{P}_i)$ for each $i<n$. We will show that the family $\{X_i : i<n\}$ of Franklin compact spaces is the desired family. Since $\mathcal{P}=\bigcup\{\mathcal{P}_i : i<n\}$ is an infinite maximal disjoint family of clopen subsets of ω^*, it follows from Lemma 2 that $\prod\{X_i : i<n\}$ is not Fréchet. So it remains to show the property (a) of our theorem.

Let $k<n$ and $\{Y_i : i<k\}$ be a family satisfying that each Y_i is equal to X_j for some $j<n$. We show that $Y=\prod\{Y_i : i<k\}$ is Fréchet. Since $k<n$, we can find $m<n$ such that $X_m\neq Y_i$ for each $i<k$. Fix such m. To show the Fréchet property of Y, let $y\in Y$, $B\subset Y$ and $y\in\operatorname{cl}B$. Without loss of generality, we may assume that $y=(\{I_0^*\}, \{I_1^*\}, \cdots, \{I_{p-1}^*\}, \infty_p, \cdots, \infty_{p+q-1}, \infty_{p+q}, \cdots, \infty_{k-1})$ and $B\subset\omega^p\times\mathcal{P}_p\times\cdots\times\mathcal{P}_{p+q-1}\times\omega^{k-p-q}$. The other cases are trivial or reducible to a case similar to this one.

Since every Franklin compact space has countable tightness, by Lemma 3, Y also has countable tightness. So we may assume that B is a countable set. Then there is a countable subfamily $Q_{p+r}\subset\mathcal{P}_{p+r}$ for each $r=0, 1, 2, \cdots, q-1$ such that $B\subset\omega^p\times Q_p\times\cdots\times Q_{p+q-1}\times\omega^{k-p-q}$. Pick an arbitrary member $I_{p+r}^*\in\mathcal{P}_{p+r}$ for each $r=0, 1, \cdots, q-1$. Since $\{\infty_{p+r}\}\cup Q_{p+r}$ is homeomorphic to a convergent sequence, it is homeomorphic to $\{I_{p+r}^*\}\cup I_{p+r}\subset\{I_{p+r}^*\}\cup\omega$. Hence we may assume moreover that $y=(\{I_0^*\}, \{I_1^*\}, \cdots, \{I_{p-1}^*\}, \{I_p^*\}, \cdots, \{I_{p+q-1}^*\}, \infty_{p+q}, \cdots, \infty_{k-1})$ and $B\subset\omega^p\times\omega^q\times\omega^{k-p-q}=\omega^k$.

By the argument similar to [2], we obtain a set $J\subset k$, a family $\{I_j : j\in J\}\subset\mathcal{J}$ and a subset A of B satisfying the assumption of Lemma 4 (c). Recall that $X_m\neq Y_i$ for each $i<k$. Therefore the family of all sets of the form $(\prod\{\omega-E(i) : i\in k-J\})\times(\prod\{I_j-F(j) : j\in J\})$, where $E(i)$ is a finite union of members of $\mathcal{J}-\mathcal{J}_m$ and $F(j)$ is a finite subset of ω, becomes a local network at y in Y. Hence the sequence $\{a_0, a_1, \cdots\}$ in A obtained by Lemma 4 (c) converges to y. The proof is completed.

Theorem 6 (MA). *For each n with $3\leqq n\leqq\omega$, there is a compact Fréchet space X such that X^k is Fréchet for any $k<n$, but X^n is not Fréchet.*

Proof. Take the family $\{X_i : i<n\}$ of Lemma 5. If $n<\omega$, let X be the disjoint topological sum $\oplus\{X_i : i<n\}$. If $n=\omega$, let X be the one point compactification of $\oplus\{X_i : i<n=\omega\}$. Since X^n contains $\prod\{X_i : i<n\}$, X^n is not Fréchet. It remains to prove that X^k is Fréchet for any $k<n$. If $n<\omega$, this follows immediately from Lemma 5.

Suppose $n=\omega$. Denote by p the only one point of the set $X-\oplus\{X_i : i<\omega\}$. We must show the Fréchet property at points of X^k whose coordinates contain p. But the neighborhood base at p in X is the same as a convergent sequence if we identify each X_i to a point x_i. Therefore by the argument similar to the proof of Lemma 5, we can show that X^k is Fréchet.

Problem. Are Lemma 5 and Theorem 6 true within ZFC?

References

[1] T. K. Boehme and M. Rosenfeld: An example of two compact Hausdorff Fréchet spaces whose product is not Fréchet. J. London Math. Soc., **8**, 339–344 (1974).
[2] G. Gruenhage: A note on the product of Fréchet spaces. Topology Proceedings, **3**, 109–115 (1978).
[3] V. I. Malyhin: On tightness and Suslin number in $\exp X$ and in a product of spaces. Soviet Math. Dokl., **13**, 496–499 (1972).
[4] V. I. Malyhin and B. E. Šapirovskiĭ: Martin's axiom and properties of topological spaces. ibid., **14**, 1746–1751 (1973).
[5] E. Michael: A quintuple quotient quest. Gen. Topology Appl., **2**, 91–138 (1972).
[6] T. Nogura: The product of $\langle \alpha_i \rangle$-spaces. Topology Appl., **21**, 251–259 (1985).
[7] R. C. Olson: Bi-quotient maps, countably bi-sequential spaces, and related topics. Gen. Topology Appl., **4**, 1–28 (1974).
[8] P. Simon: A compact Fréchet space whose square is not Fréchet. Comment. Math. Univ. Carolinae, **21**, 749–753 (1980).