84. *Some New Two-step Integration Methods*

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1. Introduction. The purpose of this is to present some new two-step methods, which deal with the following initial value problem:

(1.1) $$y'=f(x,y),\quad y(x_0)=y_0.$$

Of all computational methods for (1.1), Runge-Kutta (abbr., R-K) are most popular. R-K methods retain the advantage of one-step methods, but need some functional evaluations for each step. We shall look for other methods to decrease the functional evaluations in R-K methods. Such methods have been discussed by Byrne, Lambert [1] and many others. We have seen in [1] that two-step R-K methods have order $p(r)=r+1$ $(r=2,3,4)$, and that R-K methods [2], [3] have order $p(r)=r$ $(r=1,2,3,4)$, $p(5)=4$, $p(6)=5$, $p(r)=6$ $(r=7,8)$, $p(r)=7$ $(r=9,10)$, $p(11)=8$, where $p(r)$ denotes the highest order that can be attained by an r-stage. Thus two-step R-K methods attain higher order than R-K methods for the same stage. However, in actual computation, two-step R-K methods would not yield as good numerical results as R-K methods for the same order, and some people seem to have the opinion that two-step R-K methods may not be useful for actual computations, but some useful two-step methods are still required in many fields. We now propose the following two-step R-K methods which improve the defect of the usual two-step R-K methods:

(1.2)
$$\begin{aligned}
&y_{n+1}=V_1^{(1)}y_{n-1}+V_2^{(1)}y_n+h\Phi^{(1)}(x_{n-1},x_n,y_{n-1},y_{n-1+\theta_1},y_n,y_{n+\theta_2};h),\\
&y_{n+1+\theta}=V_1^{(2)}y_{n-1}+V_2^{(2)}y_n+h\Phi^{(2)}(x_{n-1},x_n,y_{n-1},y_{n-1+\theta_1},y_n,y_{n+\theta_2};h),\\
&\Phi^{(j)}(x_{n-1},x_n,y_{n-1},y_{n-1+\theta_1},y_n,y_{n+\theta_2})=\textstyle\sum_{i=1}^{r}(W_i^{(j)}k_i(x_{n-1})+S_i^{(j)}k_i(x_n))\\
&\qquad\qquad (0\le\theta=\theta_1,\ \theta_2\le 1),\quad (j=1,2),\\
&k_1(x_{n-j})=f(x_{n-j},y_{n-j})\quad (j=0,1),\\
&k_i(x_{n-1})=f(x_{n-1}+a_ih,\ y_{n-1}+b_iy_{n-1+\theta_1}+h\textstyle\sum_{j=1}^{r-1}b_{ij}k_j(x_{n-1})),\\
&k_i(x_n)=f(x_n+c_ih,\ y_n+d_iy_{n+\theta_2}+h\textstyle\sum_{j=1}^{r-1}d_{ij}k_j(x_n)),\\
&a_i=b_i+\textstyle\sum_{j=1}^{r-1}b_{ij},\quad c_i=d_i+\sum_{j=1}^{r-1}d_{ij}\quad (0<a_i,\ c_i\le 1).
\end{aligned}$$

In our methods, we have $p(2)=5$. In using our method, we assume that we have already computed the value of $y(x_0+\theta h)$, $y(x_0+h)$ and $y(x_0+(1+\theta)h)$ by some other means, where $y(x)$ denotes the analytical solutions of (1.1). We first calculate the value of y_1 and $y_{1+\theta_1}$ by some means of (1.2), and next proceed to the calculation of y_2 and $y_{2+\theta_2}$. To demonstrate

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our idea, we present our method (1.2) for $r=2$ below. Stability analysis, numerical results and other related results will appear elsewhere.

2. Numerical method ($r=2$). It can be seen that in (1.2) there are some parameters which must be determined. To obtain specific values for these parameters, we expand y_{n+1} in (1.2) in terms of h such as that it agrees with the solution of the differential equation up to order five in its Taylor series. This yields the following results:

$$y_{2m+1+ia}=V_1^{(i+1)}y_{2m}+V_2^{(i+1)}y_{2m+1}+h\sum_{j=1}^{r-1}(W_j^{(i+1)}k_j(x_{2m-1})+S_j^{(i+1)}k_j(x_{2m})),$$
$$k_1(x_{2m-1})=f(x_{2m-1}, y_{2m-1}),\quad k_2(x_{2m-1})=f(x_{2m-1}+ah, y_{2m-1+a}),$$
$$k_1(x_{2m})=f(x_{2m}, y_{2m}),\quad k_2(x_{2m})=f(x_{2m}+ch, y_{2m+c})\quad (i=0, 1), \tag{2.1}$$

and

$$y_{2m+2+ic}=\tilde{V}_1^{(i+1)}y_{2m+1}+\tilde{V}_2^{(i+1)}y_{2m+2}+h\sum_{j=1}^{r-1}(\tilde{W}_j^{(i+1)}k_j(x_{2m})+\tilde{S}_j^{(i+1)}k_j(x_{2m+1})),$$
$$k_1(x_{2m})=f(x_{2m}, y_{2m}),\quad k_2(x_{2m})=f(x_{2m}+ch, y_{2m+c}),$$
$$k_2(x_{2m+1})=f(x_{2m+1}, y_{2m+1}),\quad k_2(x_{2m+1})=f(x_{2m+1}+ah, y_{2m+1+a})\quad (i=0, 1),$$

where

$$W_i^{(1)}=W_i(a, c, 0),\quad S_i^{(1)}=S_i(a, c, 0),\quad V_i^{(1)}=V_i(a, c, 0),$$
$$W_i^{(2)}=W_i(a, c, a),\quad S_i^{(2)}=S_i(a, c, a),\quad V_i^{(2)}=V_i(a, c, a),$$
$$\tilde{W}_i^{(1)}=W_i(c, a, 0),\quad \tilde{S}_i^{(1)}=S_i(c, a, 0),\quad \tilde{V}_i^{(1)}=V_i(c, a, 0),$$
$$\tilde{W}_i^{(2)}=W_i(c, a, a),\quad \tilde{S}_i^{(2)}=S_i(c, a, 0),\quad \tilde{V}_i^{(2)}=V_i(c, a, 0),$$
$$V_2^{(i)}=1-V_1^{(i)},\quad \tilde{V}_2^{(i)}=1-\tilde{V}_1^{(i)}\quad (i=1, 2),$$
$$S_2(a, b, \theta)(=S_2)=(\theta+1)^2(\theta+2)^2\{(a-1)(5a-2)-(4a-2)(\theta+1)\}$$
$$\times\{(2b(b+1)(10ab+5a-5b-2)(a-b-1)\}^{-1},$$
$$V_1(a, b, \theta)(=V_1)=[b^2(b+1)(a-b-1)S_2-(1/60)(\theta+1)^3$$
$$\times\{5(3\theta+7)(a-1)-3(\theta+1)(4\theta+9)\}]60(5a-2)^{-1},$$
$$W_2(a, b, \theta)(=W_2)=\{(\theta+1)^2(2\theta+5)-6b(b+1)S_2-V_1\}\{6a(a-1)\}^{-1},$$
$$W_1(a, b, \theta)(=W_1)=-0.5(\theta+1)^2+0.5V_1+(a-1)W_2+bS_2,$$
$$S_1(a, b, \theta)=1+\theta-(W_1+W_2+S_2-V_1).$$

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References

[1] G. D. Byrne and R. J. Lambert: Pseudo-Runge-Kutta methods involving two points. J. A. C. M., **13**, 114–123 (1966).

[2] J. C. Butcher: On the attainable order of Runge-Kutta methods. Math. Comp., **19**, 408–417 (1965).

[3] ——: The non-existence of the stage eight order explicit Runge-Kutta methods. BIT., **25**, 521–540 (1985).

[4] C. W. Gear: Runge-Kutta starters for multistep methods. A. C. M. Trans. Math. Software, **6**, 263–279 (1980).

[5] M. Nakashima: On a Pseudo-Runge-Kutta methods with 2 and 3 stages. Publ. RIMS, Kyoto Univ., **18**, 895–909 (1982).

[6] M. Tanaka: Pseudo-Runge-Kutta Methods and their application to the estimation

of truncation error in 2nd and 3rd order Runge-Kutta methods. Joho Shori., **10**(6), 406–417 (1969).

[7] B. P. Sommeijer and P. J. Van der Houwen: On the economization of stabilized Runge-Kutta methods with application to parabolic initial value problems. Report NW **74**, **75**, Mathematisch Centrum, Amsterdam, 1979.

[8] P. J. Van der Houwen: Construction of integration formulas for initial problems. Amsterdam: North-Holland Publishing Company (1976).