80. *Large-time Behavior of Solutions for Hyperbolic-parabolic Systems of Conservation Laws*

By Shuichi KAWASHIMA
Department of Mathematics, Nara Women's University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 13, 1986)

1. Introduction. We consider the system of conservation equations

(1) $$f^0(u)_t + f(u)_x = (G(u)u_x)_x, \quad t \geq 0, \quad x \in \boldsymbol{R},$$

where $u = u(t, x)$ is an m-vector, $f^0(u)$ and $f(u)$ are smooth m-vector valued functions, and $G(u)$ is a smooth $m \times m$ matrix. We assume that $Df^0(u)$, the Jacobian of $f^0(u)$, is non-singular and the mapping $v = f^0(u)$ is one-to-one so that (1) is equivalent to

(2) $$v_t + f(u(v))_x = (\tilde{B}(u(v))v_x)_x, \qquad v = f^0(u).$$

Here $u = u(v)$ is the inverse mapping of $v = f^0(u)$ and $\tilde{B}(u) = G(u)Df^0(u)^{-1}$. We study the large-time behavior of solution of (1). It is shown that as $t \to \infty$, the solution of (1) approaches the superposition of the nonlinear and linear diffusion waves constructed by solutions of the Burgers equation and the linear heat equation. The same problem was discussed in [5] for a model system of a viscous gas.

2. Global existence and decay of solutions. As the first step, we consider the global existence problem for (1). This problem has been solved in [2] under the conditions (i)–(iii) described below.

(i) The system (1) has a strictly convex entropy ([1], [2]).

This condition enables us to reduce the system (1) to the symmetric form

(3) $$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x).$$

Here $A^0(u)$, $A(u)$ and $B(u)$ are $m \times m$ symmetric matrices such that $A^0(u)$ is positive definite and $B(u)$ is nonnegative definite. For the explicit form of (3), see [1], [2].

(ii) The associated symmetric system (3) is hyperbolic-parabolic ([2]).

(iii) The linearized system of (3) around a given constant state $u = \bar{u}$ satisfies the stability condition ([6]): Let $\lambda A^0(\bar{u})\phi = A(\bar{u})\phi$ and $B(\bar{u})\phi = 0$ for $\lambda \in \boldsymbol{R}$ and $\phi \in \boldsymbol{R}^m$. Then $\phi = 0$.

The results concerning the global existence and decay of solution of (1) are summarized in the following theorem.

Theorem 1 ([2]). *Let $\bar{u}$ be a constant state and assume* (i)–(iii). *Consider* (1) *with the initial data $u(0, x) = u_0(x)$. If $u_0(x) - \bar{u}$ is small in H^s, $s \geq 2$, then* (1) *has a unique global solution $u(t, x)$ which converges to $\bar{u}$ uniformly in $x \in \boldsymbol{R}$ as $t \to \infty$. If, in addition, $u_0(x) - \bar{u}$ is small in $H^s \cap L^1$, $s \geq 3$, then the L^2-norm of $\partial_x^l(f^0(u(t, x)) - f^0(\bar{u}))$ tends to zero at the rate $t^{-(1/2+l)/2}$ as $t \to \infty$, where $3l \leq s - 2$.*

The first part of the theorem is proved by an energy method which makes use of the strict convexity of the entropy. The second part follows from the conservation form (2) and the decay estimate for the semigroup e^{tR} associated with the linearized system of (2) around $v=f^0(\bar{u})$:

$$\|\partial_x^l(e^{tR}v)\| \leq Ce^{-ct}\|\partial_x^l v\| + C(1+t)^{-(1/2+l-k)/2}\|\partial_x^k v\|_{L^1}, \tag{4}$$

where $\|\cdot\|$ denotes the L^2-norm, $k \leq l$, and c and C are positive constants ([2], [6]).

3. Approximation by uniformly parabolic system. The second step is to approximate (2) by a uniformly parabolic system. To this end we require:

(iv) The associated inviscid system $v_t+f(u(v))_x=0$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate ([3]).

Put $\tilde{A}(u)=Df(u)Df^0(u)^{-1}$. Notice that $\tilde{A}(u(v))$ is the Jacobian of $f(u(v))$ with respect to v. We denote the eigenvalues and right eigenvectors of $\tilde{A}(u)$ by $\lambda_j(u)$ and $\tilde{r}_j(u)$, $j=1, \cdots, m$, respectively. Let $\tilde{P}_j(u)$, $j=1, \cdots, m$, be the corresponding eigenprojections and put

$$\tilde{D}(u)=\sum_{j=1}^{m}\kappa_j(u)\tilde{P}_j(u), \qquad \kappa_j(u)=\frac{\langle B(u)\hat{r}_j(u), \hat{r}_j(u)\rangle}{\langle A^0(u)\hat{r}_j(u), \hat{r}_j(u)\rangle}, \tag{5}$$

where $\hat{r}_j(u)=Df^0(u)^{-1}\tilde{r}_j(u)$ and $\langle\ ,\ \rangle$ denotes the inner product of $\boldsymbol{R}^m$. We then consider the system

$$w_t+f(u(w))_x=\tilde{D}(\bar{u})w_{xx}. \tag{6}$$

This is a uniformly parabolic system because $\kappa_j(\bar{u})>0$, $j=1, \cdots, m$, by (iii). We denote by e^{tS} the semigroup associated with the linearized system of (6) around $w=f^0(\bar{u})$. This semigroup has the same estimate as in (4) and the uniformly parabolic system (6) is solved globally in time as in Theorem 1. Also, it is shown that $f^0(u(t,x))$ is well approximated by the solution $w(t,x)$ of (6), namely, we have the following

Theorem 2. *Assume* (i)–(iv). *Let $u_0(x)-\bar{u}$ be small in $H^s\cap L^1$, $s\geq 5$, and let $w(t,x)$ be the solution of* (6) *with the initial data $w(0,x)=f^0(u_0(x))$. Then the L^2-norm of the difference $\partial_x^l(f^0(u(t,x))-w(t,x))$ tends to zero at the rate $t^{-(3/2+l)/2+\alpha}$ as $t\to\infty$, where $3l\leq s-5$ and $\alpha>0$ is a small fixed constant.*

This theorem follows from the following better decay estimate for the difference $e^{tR}-e^{tS}$.

$$\|\partial_x^l(e^{tR}-e^{tS})v\| \leq Ce^{-ct}\|\partial_x^l v\| + C(1+t)^{-(3/2+l-k)/2}\|\partial_x^k v\|_{L^1}. \tag{7}$$

4. Large-time behavior of solutions. The final step is to construct an asymptotic solution of the uniformly parabolic system (6). This step has been well studied by Liu [4]. We determine $\delta=(\delta_1, \cdots, \delta_m)$ by

$$\int (f^0(u_0(x))-f^0(\bar{u}))dx=\sum_{j=1}^{m}\delta_j\tilde{r}_j(\bar{u}). \tag{8}$$

Following [4], we define $\bar{v}(t,x)$, the superposition of the diffusion waves, by

$$\bar{v}(t,x)-f^0(\bar{u})=\sum_{j=1}^{m}(\bar{v}^j(t,x)-f^0(\bar{u})). \tag{9}$$

Each diffusion wave $\bar{v}^j(t, x)$ is constructed as follows: When j-th characteristic field is genuinely nonlinear, $\bar{v}^j(t, x)$ lies on the integral curve of $\tilde{r}_j(u(v))$ through $v=f^0(\bar{u})$ and satisfies $\lambda_j(u(\bar{v}^j(t, x)))-\lambda_j(\bar{u})=\bar{z}^j(t+1, x-\lambda_j(\bar{u})(t+1))$. Here $\bar{z}^j(t, x)$ is the self-similar solution of the Burgers equation $z_t+zz_x=\kappa_j(\bar{u})z_{xx}$ with the property

$$\int \bar{z}^j(t, x)dx=\delta_j.$$

When j-th characteristic field is linearly degenerate, $\bar{v}^j(t, x)$ is defined similarly by using the self-similar solution of the linear heat equation $z_t=\kappa_j(\bar{u})z_{xx}$.

We employ the technique of Liu [4] and construct the linear hyperbolic wave $\zeta(t, x)$ by which $w(t, x)-\bar{v}(t, x)-\zeta(t, x)$ has zero integral for each $t\geq 0$. By virtue of this property we get the following theorems.

Theorem 3. *Assume* (i)–(iv). *Let $u_0(x)-\bar{u}$ be small in $H^s\cap L^1_\beta$, $s\geq 1$, $0<\beta<1/2$, and let $\bar{v}(t, x)$ be the superposition of the diffusion waves. Then the L^2-norm of the difference $\partial_x^l(w(t, x)-\bar{v}(t, x))$ tends to zero at the rate $t^{-(1+l)/2+\alpha}$ as $t\to\infty$, where $l\leq s$ and $\alpha=(1/2-\beta)/2$.*

Theorem 4. *Assume the same conditions of Theorem 3 for $s\geq 5$. Then the L^2-norm of the difference $\partial_x^l(f^0(u(t, x)-\bar{v}(t, x)))$ tends to zero at the rate $t^{-(1+l)/2+\alpha}$ as $t\to\infty$, where $3l\leq s-5$ and $\alpha=(1/2-\beta)/2$.*

The last theorem is a consequence of Theorems 2 and 3. When $|\delta|\neq 0$, Theorems 2, 3 and 4 give meaningful asymptotic relations as $t\to\infty$, because for large t, the L^2-norm of $\partial_x^l(\bar{v}(t, x)-f^0(\bar{u}))$ is bounded from below by $c|\delta|t^{-(1/2+l)/2}$ with a positive constant c. Finally, we remark that our results are applicable to the equations of viscous (or inviscid) heat-conductive fluids.

References

[1] K. O. Friedrichs and P. D. Lax: Systems of conservation equations with a convex extention. Proc. Nat. Acad. Sci. USA, **68**, 1686–1688 (1971).

[2] S. Kawashima: Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. Doctoral Thesis, Kyoto University, 1983.

[3] P. D. Lax: Hyperbolic systems of conservation laws, II. Comm. Pure Appl. Math., **10**, 537–566 (1957).

[4] T.-P. Liu: Nonlinear stability of shock waves for viscous conservation laws. Memoirs, Amer. Math. Soc., no. 328, vol. 56 (1985).

[5] T. Nishida: Equations of motion of compressible viscous fluids (to appear).

[6] Y. Shizuta and S. Kawashima: Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. Hokkaido Math. J., **14**, 249–275 (1985).