76. On Cusp Forms of Octahedral Type

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1. In [1], Chinburg formulated Stark conjecture "over $\boldsymbol{Z}$". He proved for one dimensional case and gave several examples for two dimensional representations of tetrahedral type. The purpose of this note is to give an example of octahedral type.

First recall some notations of [1]. Let K be a finite Galois extension of $\boldsymbol{Q}$ with Galois group G. Let S_∞ be the set of infinite places of K and let $\| \ \|_v$ be the normalized absolute value for $v \in S_\infty$. Let $\boldsymbol{V}=(V, S)$ denote the pair of an irreducible complex representation V of G and a finite set S of places of $\boldsymbol{Q}$ containing ∞. Define $L(s, \boldsymbol{V})=\prod_{p \notin S} L_p(s)$ where $L_p(s)$ is a usual Euler p-factor. Let χ_V be the character of V and put $\mathrm{pr}_V=\sum_{g\in G}\chi_V(g)g$. Let $L_{\boldsymbol{V}}(s)=L(s, \boldsymbol{V})\,\mathrm{pr}_V$ and $L'_{\boldsymbol{V}}(s)=L'(s, \boldsymbol{V})\,\mathrm{pr}_V$. We only consider representations V such that $L(s, \boldsymbol{V})$ has a zero of order 1 at $s=0$.

Let $d=\sum d_{\boldsymbol{V}}\boldsymbol{V}$ be a finite linear combination of $\boldsymbol{V}$ of dimension n. We define $L(s, d)=\sum d_{\boldsymbol{V}}L(s, \boldsymbol{V})$, $L'(s, d)=\sum d_{\boldsymbol{V}}L'(s, \boldsymbol{V})$, $L_d(s)=\sum d_{\boldsymbol{V}}L_{\boldsymbol{V}}(s)$ and $L'_d(s)=\sum d_{\boldsymbol{V}}L'_{\boldsymbol{V}}(s)$. The element $\tau\in \mathrm{Aut}\,(\boldsymbol{C}/\boldsymbol{Q})$ acts on d by $d^\tau=\sum d^\tau_{\boldsymbol{V}}\boldsymbol{V}^\tau$ where $\boldsymbol{V}^\tau=(V^\tau, S)$ for $\boldsymbol{V}=(V, S)$. We define the additive group $\boldsymbol{D}(n)$ by the set of all d such that $d^\tau=d$ for any $\tau\in\mathrm{Aut}\,(\boldsymbol{C}/\boldsymbol{Q})$ and such that $L(s, d)$ is a Dirichlet series with integral coefficients.

Stark Conjecture. Suppose that $n=1$ or 2, and $d\in \boldsymbol{D}(n)$. Then $\exp\,(L'(0, d))=e(d)$ is a real unit in K, and $L'_d(0)v_0=\sum_{v\in S_\infty}\log\|e(d)\|_v\, v$ where v_0 is a fixed embedding of K into $\boldsymbol{C}$. If $n=2$, the real conjugates of $e(d)$ are positive.

2. From now on, we consider the space of cusp forms of weight 1 on $\Gamma_0(283)$ with the character $(-283/*)$. This space has one primitive form of S_3 type and two primitive forms of S_4 type (cf. Serre [2]). Let h be of S_3 type and let f be one of the forms of S_4 type. Then the other form of S_4 type is f^ρ where ρ is a complex conjugation. The Fourier coefficients of h and f are listed in the table below. Numerical computation shows that

$$L'(0, f)=2.4681497509\cdots+0.2223673138\cdots i,$$
$$L'(0, h)=2.802684\cdots.$$

Let V_f and W be Galois representations attached to f and h respectively. We denote by φ the projection $GL_2(\boldsymbol{C})\to PGL_2(\boldsymbol{C})$, and put $\tilde{V}_f=\varphi\cdot V_f$ and $\tilde{W}=\varphi\cdot W$. Let K and L be the fields corresponding to the kernels of V_f and $\tilde{V}_f$ respectively. Then

$$\boldsymbol{Q}(\sqrt{-283})\subset H\ (=\text{the absolute class field of } \boldsymbol{Q}(\sqrt{-283}))\subset L,$$

and H is the field corresponding to the kernel of W. We have $\mathrm{Gal}(L/\boldsymbol{Q})\simeq S_4$ and $\mathrm{Gal}(H/\boldsymbol{Q})\simeq S_3$.

These fields are constructed explicitly as follows. Let $x^3+4x-1=(x-\alpha)(x-\beta_1)(x-\beta_2)$ where $\alpha\in\boldsymbol{R}$, and $\beta_2=\bar{\beta}_1$. Then we have $H=\boldsymbol{Q}(\alpha,\beta_1,\beta_2)$ and $L=\boldsymbol{Q}(\sqrt{\alpha},\sqrt{\beta_1},\sqrt{\beta_2})$. Put $\gamma=\sqrt{\beta_1}+\sqrt{\beta_2}$, then $L_+=\boldsymbol{Q}(\gamma)$ is the maximal real subfield of L and corresponds to the centralizer of ρ in $\mathrm{Gal}(K/\boldsymbol{Q})$. The integral basis of L_+ is given by

$$\begin{aligned}
&\omega_1=1, && \omega_2=\alpha, && \omega_3=\alpha^2,\\
&\omega_4=(1+\alpha\sqrt{\alpha})/2, && \omega_5=(\alpha^2+\sqrt{\alpha})/2, && \omega_6=(\alpha+\alpha^2\sqrt{\alpha})/2,\\
&\omega_7=(\omega_1+\omega_2\gamma)/2, && \omega_8=(\omega_2+\omega_3\gamma)/2, && \omega_9=(\omega_3+\omega_1\gamma)/2,\\
&\omega_{10}=(\omega_4+\omega_6\gamma)/2, && \omega_{11}=(\omega_5+\omega_4\gamma)/2, && \omega_{12}=(\omega_6+\omega_5\gamma)/2.
\end{aligned}$$

3. Let G be a subgroup of $GL_2(\boldsymbol{C})$ generated by

$$g_1=\begin{pmatrix}0, & -(1+i)/\sqrt{2}\\ -(1-i)/\sqrt{2}, & 0\end{pmatrix},\quad g_2=\begin{pmatrix}-1/\sqrt{2}, & -i/\sqrt{2}\\ i/\sqrt{2}, & 1/\sqrt{2}\end{pmatrix}\quad\text{and}$$

$$g_3=\begin{pmatrix}-1/\sqrt{2}, & i/\sqrt{2}\\ -i/\sqrt{2}, & 1/\sqrt{2}\end{pmatrix}.$$

G is of order 48 and is isomorphic to $\mathrm{Gal}(K/\boldsymbol{Q})$. Let $\langle g\rangle$ denote the conjugacy class containing g, and let E_2 denote the unit matrix of size 2. Then the conjugacy classes of G are $\langle E_2\rangle$, $\langle -E_2\rangle$, $\langle g_2g_3\rangle$, $\langle g_1g_2\rangle$, $\langle -g_1g_2\rangle$, $\langle g_1\rangle$, $\langle g_1g_2g_3\rangle$ and $\langle -g_1g_2g_3\rangle$. G has three irreducible two dimensional representations: the natural representation V, V^ρ and the one which is obtained from the two dimensional representation of S_3 via the map $G\to S_4\to S_3$. The last one is isomorphic to W introduced above. So we denote it by the same letter W.

The isomorphism of $\varphi(G)$ and S_4 is given by $\varphi(g_1)=(1\ 2)$, $\varphi(g_2)=(1\ 3)$ and $\varphi(g_3)=(2\ 4)$. Taking the conjugates if necessary, we can set that g_1 is the complex conjugation ρ in $\mathrm{Gal}(K/\boldsymbol{Q})$, and $\sqrt{\alpha}^{(1\,2\,3)}=\sqrt{\beta_1}$, $\sqrt{\alpha}^{(1\,3\,2)}=\sqrt{\beta_2}$. Hereafter we fix this identification.

4. We write V for $(V,\{\infty\})$ for simplicity. Let $L'_{\boldsymbol{D}(2)}(0)=\{L'_d(0)\mid d\in\boldsymbol{D}(2)\}$. This set is equal to the set consisting of $L'_d(0)$ when d runs through the subgroup generated by $\delta V+\delta^\rho V^\rho$ for $\delta\in D^{-1}_{\boldsymbol{Q}(\sqrt{2}i)}$, W and $(1/4)(V+V^\rho)+(1/2)W$. Here D_k is the different of the field k.

Theorem. *Suppose that the Stark Conjecture is true.*

(i) *Let* $d=\delta V+\delta^\rho V^\rho$ *for* $\delta\in D^{-1}_{\boldsymbol{Q}(\sqrt{2}i)}$. *Then the equation of* $e(d)$ *over* L_+ *is*

$$x^2-t(\delta)x+1=0$$

where

$$t(1/2)=2\omega_1+3\omega_2-8\omega_3-\omega_4+13\omega_5-4\omega_6-2\omega_7-6\omega_8+3\omega_9+3\omega_{10}+\omega_{11}+14\omega_{12},$$

and

$$t(1/2\sqrt{2}\,i)=\omega_1+3\omega_2+\omega_3-\omega_4-\omega_5-6\omega_6-\omega_7-5\omega_8+2\omega_{10}-\omega_{11}+10\omega_{12}.$$

(ii) *Let* $d=W$. *Then* $e(W)=16\omega_1+\omega_2+4\omega_3\in\boldsymbol{Q}(\alpha)$, *and its equation over* $\boldsymbol{Q}$ *is*

$$x^3-16x^2-8x-1=0.$$

(iii) *Let* $d=(1/4)(V+V^{\rho})+(1/2)W$. *Then the equation of* $e(d)$ *over* L_+ *is*

$$x^2-sx+e(W)=0$$

where

$$s=3\omega_1-11\omega_3+20\omega_5-\omega_6-\omega_7-5\omega_8+3\omega_9+3\omega_{10}+\omega_{11}+12\omega_{12}.$$

The proof is done by the same way as in [1]. But we must make one remark. Under the above identification of $\varphi(G)$, S_4 and $\mathrm{Gal}(L/\boldsymbol{Q})$, the natural representation V is isomorphic to V_f or $V_{f\rho}$. If we assume that $V \simeq V_{f\rho}$, there is no integer in L_+ corresponding to

$$e\left(\frac{1}{2}(V+V^{\rho})\right)+e\left(-\frac{1}{2}(V+V^{\rho})\right).$$

Hence we have $V \simeq V_f$.

References

[1] T. Chinburg: Stark's conjecture for L-functions with first-order zeros at $s=0$. Adv. in Math., **48**, 82–113 (1983).

[2] J.-P. Serre: Modular forms of weight one and Galois representations. Algebraic Number Fields (ed. by A. Fröhlich), Academic Press, 193–268 (1977).

[3] H. M. Stark: Class fields and modular forms of weight one. Lect. Notes in Math., Springer, no. 601, 277–287 (1977).

Table. p-th Fourier coefficients of f and h

p	f	h	p	f	h	p	f	h	p	f	h
1	1	1	149	$-\sqrt{2}i$	0	349	-1	-1	571	0	0
2	$-\sqrt{2}i$	0	151	-1	-1	353	1	-1	577	0	2
3	$\sqrt{2}i$	0	157	-1	-1	359	0	0	587	0	0
5	$\sqrt{2}i$	0	163	-1	-1	367	0	0	593	$\sqrt{2}i$	0
7	-1	-1	167	$\sqrt{2}i$	0	373	-1	-1	599	0	0
11	1	-1	173	$-\sqrt{2}i$	0	379	1	-1	601	$\sqrt{2}i$	0
13	1	-1	179	1	-1	383	0	2	607	0	2
17	0	0	181	0	2	389	-1	-1	613	$-\sqrt{2}i$	0
19	$-\sqrt{2}i$	0	191	0	0	397	$\sqrt{2}i$	0	617	-1	-1
23	-1	-1	193	0	0	401	0	0	619	0	0
29	-1	-1	197	0	0	409	0	0	631	0	0
31	$-\sqrt{2}i$	0	199	1	-1	419	-1	-1	641	0	0
37	0	0	211	-1	-1	421	1	-1	643	2	2
41	1	-1	223	$\sqrt{2}i$	0	431	$-\sqrt{2}i$	0	647	1	-1
43	$-\sqrt{2}i$	0	227	0	2	433	1	-1	653	$-\sqrt{2}i$	0
47	$\sqrt{2}i$	0	229	0	0	439	$\sqrt{2}i$	0	659	-1	-1
53	0	0	233	-1	-1	443	1	-1	661	-1	-1
59	1	-1	239	0	0	449	$\sqrt{2}i$	0	673	0	0
61	1	-1	241	0	0	457	-1	-1	677	-1	-1
67	0	0	251	1	-1	461	$\sqrt{2}i$	0	683	0	2
71	0	2	257	-1	-1	463	0	0	691	0	0
73	0	2	263	-1	-1	467	$-\sqrt{2}i$	0	701	1	-1
79	0	0	269	-1	-1	479	0	2	709	0	2
83	-2	2	271	1	-1	487	-1	-1	719	0	0
89	-1	-1	277	$-\sqrt{2}i$	0	491	0	2	727	0	2
97	1	-1	281	0	2	499	1	-1	733	0	0
101	0	2	283	1	1	503	0	0	739	$\sqrt{2}i$	0
103	-1	-1	293	1	-1	509	0	0	743	0	0
107	0	0	307	1	-1	521	-1	-1	751	-1	-1
109	$\sqrt{2}i$	0	311	0	2	523	1	-1	757	0	0
113	0	2	313	$-\sqrt{2}i$	0	541	$-\sqrt{2}i$	0	761	1	-1
127	0	2	317	1	-1	547	1	-1	769	0	2
131	0	0	331	0	0	557	0	0	773	2	2
137	1	-1	337	1	-1	563	-1	-1	787	$-\sqrt{2}i$	0
139	$\sqrt{2}i$	0	347	0	2	569	0	0	797	$-\sqrt{2}i$	0