75. A Problem of Orlicz in the Scottish Book

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Orlicz posed the following problem in the Scottish Book ([1], [2]).

Orlicz's Problem (No. 121). *Give an example of a trigonometric series*

$$\sum (a_n \cos nx + b_n \sin nx) \tag{1}$$

everywhere divergent and such that

$$\sum (|a_n|^{2+\varepsilon} + |b_n|^{2+\varepsilon}) < \infty \tag{2}$$

for every $\varepsilon > 0$.

Our main purpose is to give an example in answer to this problem.

The problem was studied by Banach and his colleagues, especially by Orlicz ([3]). It should be worth noticing that the real difficulty or spirit of the problem lies in the point that Orlicz asked a concrete example instead of a mere existence proposition, and also in the point that the example must furnish the property of everywhere divergence.

First of all, we shall mention some partial answers applying known results on the size of certain partial sums of the exponential series

$$\sum c_n e^{2\pi i n x}. \tag{3}$$

Proposition 1. *Set* $c_n = n^{-1/2} \exp(ian \log n)$ $(a>0)$. *Then* (3) *diverges for almost all* x, *whereas*

$$\sum |c_n|^{2+\varepsilon} < \infty$$

for every $\varepsilon > 0$.

This follows from M. Weiss' result ([4]). It seems that she was not aware of Orlicz's problem. By using probability method, she investigated the behavior of the exponential series

$$\sum n^{-1/2} \exp(i\beta n \log n + in\theta).$$

Proposition 2. *Set*

$$c_n = \begin{cases} k^{-1/2}(\log k)^{-1/4} & \text{if } n = k^2 > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then holds the same conclusion as in Proposition 1.

This follows from Fiedler-Jurkat-Körner's result ([5]). They were interested in the almost all estimates of the sum

$$S_N(x) = \sum_{n \leq N} e^{\pi i n^2 x} \quad (x \in \boldsymbol{R}).$$

Proposition 3. *There exists a sequence* $c_n \to 0$ *such that* (3) *diverges for all* x, *yet*

$$\sum |c_n|^{2+\varepsilon} < \infty$$

for every $\varepsilon > 0$.

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This follows from T. W. Körner's result ([6]). Körner proved, among other things, the following result: If N is sufficiently large, then there exist $a_1, a_2, \cdots, a_N (\in \boldsymbol{C})$ with $|a_n|=1$ such that for all $t \in \boldsymbol{R}$

$$\sqrt{N} \ll \sum_{1 \leqq n \leqq N} a_n e^{int} \ll \sqrt{N}.$$

However, Körner's method of proof is not effective, and cannot supply any concrete example to Orlicz's problem.

On the other hand, Hardy and Littlewood ([7]) showed, from a different point of view,

$$S(N) = \sum_{n \leqq N} \exp(2\pi i(\alpha n \log n + \beta n)) \ll \sqrt{N}$$

uniformly in α and β.

As far as we know, it does not seem to have been investigated whether it satisfies $S(N) \gg \sqrt{N}$ for all $\beta \in \boldsymbol{R}$.

We shall show the following

Theorem. *Let α be real and positive, and β real. Then*

$$|\sum_{e^{-1/\alpha}N < n \leqq N} \exp(2\pi i(\alpha n \log n + \beta n))| \geqq C_\alpha \sqrt{N} \tag{4}$$

holds for all α and β, where C_α depends at most on α.

Then we have the following

Corollary 1.

$$S_N = |\sum_{e^{-1/\alpha}N < n \leqq N} \cos(2\pi\alpha n \log n + nx)|$$

$$\geqq \frac{\sqrt{N}}{\sqrt{\alpha}} e^{-1/(2\alpha)} \left|\cos\left(2\pi\alpha N \exp\left(-\frac{\{\alpha \log N + \alpha + x\}}{\alpha}\right) - \frac{\pi}{4}\right)\right| + O(1/\alpha),$$

where $\{y\}$ denotes the fractional part of y.

Now we give an example to Orlicz's problem. Let us set

$$\begin{cases} a_n = n^{-1/2} \cos(2\pi\alpha n \log n), \\ b_n = -n^{-1/2} \sin(2\pi\alpha n \log n), \end{cases} \tag{5}$$

and we shall show that this example satisfies the desired conditions for some α. It is clear then

$$\sum (|a_n|^{2+\varepsilon} + |b_n|^{2+\varepsilon})$$

is convergent for every $\varepsilon > 0$.

Lemma. *If an infinite series*

$$\sum_{n=1}^{\infty} d_n$$

converges, then for any positive sequence $k_n \uparrow \infty$, we have

$$\sum_{AN < n \leqq N} k_n d_n = o(k_N),$$

where A is a constant $(0 \leqq A < 1)$.

Now we show that (1) diverges everywhere for certain values of α. Let us suppose that

$$\sum (a_n \cos nx + b_n \sin nx) = \sum n^{-1/2} \cos(2\pi\alpha n \log n + nx)$$

converges at some x. Then putting $k_n = \sqrt{n}$ in the lemma, we have

$$\sum_{e^{-1/\alpha}N < n \leqq N} \cos(2\pi\alpha n \log n + nx) = o(\sqrt{N})$$

for some $x \in \boldsymbol{R}$. But on the contrary it follows from Corollary 1 that

$$\limsup_{N\to\infty} \frac{S_N}{\sqrt{N}} > 0$$

for some $x \in \boldsymbol{R}$, where $\alpha=1$, $1/\log 2$, e.g. For an instance, for $\alpha=1$ we have

$$\limsup_{N\to\infty} \left|\cos\left(2\pi\alpha N \exp\left(-\frac{\{\alpha\log N+\alpha+x\}}{\alpha}\right)-\frac{\pi}{4}\right)\right|>0$$

by applying inhomogeneous Diophantine approximation. This is a contradiction. Therefore (1) must diverge everywhere for such an α.

Finally we shall give

Sketch of Proof of Theorem. Put

$$f(x)=\alpha x\log x+\beta x \qquad (\alpha>0,\ 0\leqq\beta<1).$$

Since $f'(x)$ is an increasing function, there exists a γ_k ($k \in \boldsymbol{N}$) such that

$$f'(\gamma_k)=k-1/2 \quad \text{and} \quad \gamma_k<\gamma_{k+1}.$$

Define

$$M=[\alpha\log N+\alpha+\beta],$$

where $[y]$ denotes the integral part of y, and

$$\gamma_M=e^{-1/\alpha}\gamma_{M+1}.$$

Let us set

$$S_M=\sum_{\gamma_M<n\leqq\gamma_{M+1}} e^{2\pi i f(n)}.$$

Then by van der Corput's lemmas ([8])

$$S_M=\int_0^\infty e^{2\pi i(f(x)-Mx)}dx+O(1).$$

Then by the method of the stationary phase ([9]), we obtain (4).

If we restrict the range of the values of α, then we have

Corollary 2. *If* $0<\alpha<1/(2\log 2)$, *then*

$$\sum_{1\leqq n\leqq N} \exp\left(2\pi i(\alpha n\log n+\beta n)\right)\gg\sqrt{N}$$

holds for all α *and* β, *where the implied constant depends only on* α.

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