74. Group Rings whose Augmentation Ideals are Residually Lie Solvable

By Tadashi MITSUDA
Department of Mathematics, Tokyo Metropolitan University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1986)

1. Introduction. Let R be a commutative ring with identity and G be a group. We denote the augmentation ideal of the group ring RG by $\Delta_R(G)$. There are many problems and results relating to $\Delta_R(G)$ (cf. [6]). In particular, it is an interesting problem to characterize the group rings whose augmentation ideals satisfy some conditions. In this paper, we treat the Lie property. We recall some definitions. Let S be a ring and I be a two sided ideal of S. Then $I^{\langle n\rangle}$ and $I^{(n)}$ are the ideals of S defined inductively as follows, respectively.

$$I^{\langle 1\rangle}=I, \qquad I^{\langle n+1\rangle}=[I^{\langle n\rangle}, I^{\langle n\rangle}]S$$
$$I^{(1)}=I, \qquad I^{(n+1)}=[I, I^{(n)}]S,$$

where $[M, N]$ is the additive subgroup of S generated by the elements of the form $[m, n]=mn-nm$ with $m\in M$ and $n\in N$. We say that I is *Lie solvable* (resp. *Lie nilpotent*) if $I^{\langle n\rangle}=0$ for some n (resp. $I^{(n)}=0$ for some n). And I is called *residually Lie solvable* (resp. *residually Lie nilpotent*) if $\bigcap_n I^{\langle n\rangle}=0$ (resp. $\bigcap_n I^{(n)}=0$).

Parmenter-Passi-Sehgal [5] characterizes those groups G such that $\Delta_R(G)$ is Lie nilpotent. The condition under which $\Delta_k(G)$ is residually Lie nilpotent when k is a field is also known (cf. [6]). Further, Musson-Weiss [4] gave the characterization of the groups G such that $\Delta_Z(G)$ is residually Lie nilpotent. In [7], the groups G such that RG is Lie solvable are characterized (Lie solvability in our sense is called "strong" Lie solvability in that book). On the other hand, we have $\Delta_R^{\langle n\rangle}(G)=RG^{\langle n\rangle}$ and $\Delta_R^{(n)}(G)=RG^{(n)}$ because $[x, y]=[x-\varepsilon(x)\cdot 1, y-\varepsilon(y)\cdot 1]$ where $x, y\in RG$ and ε is the augmentation map. Thus those groups G such that $\Delta_R(G)$ is Lie solvable are already characterized. Now the aim of this paper is to show the following

Theorem. *Let G be a finite group. Then $\bigcap_n \Delta_Z^{\langle n\rangle}(G)=0$ if and only if G' is a p-group for some prime p, where G' is the commutator subgroup of G.*

2. Preliminaries. The following is the key lemma to prove our theorem.

Lemma. *Let R be a commutative ring with identity and G be a finite group. Let K, L be the subgroups of G such that $K\leqq L\leqq N_G(K)$ and put $N=(K, L)=\langle k^{-1}l^{-1}kl \mid k\in K, l\in L\rangle$. Then for any $x\in N$ and $n\geqq 2$, we have*

$$(*) \qquad |N|^{2^{n-1}-2}(x-1)\in \Delta_R^{\langle n\rangle}(G).$$

Proof. We use the induction on n. Since

$$ghg^{-1}h^{-1}-1=\{(g-1)(h-1)-(h-1)(g-1)\}g^{-1}h^{-1}$$

for $g, h \in G$, we have $x-1 \in \Delta_R^{\langle 2\rangle}(G)$ for any $x \in N \leqq G'$. Assume that $(*)$ holds for $n-1$. Let $g, h \in N$, $x \in K$ and $y \in L$. Since $\Delta_R^{\langle n-1\rangle}(G)$ is an ideal, $|N|^{2^{n-2}-2}(g-1)x$ and $|N|^{2^{n-2}-2}(h-1)y$ belong to $\Delta_R^{\langle n-1\rangle}(G)$ by the induction hypothesis. Thus we have

$$\begin{aligned}&[|N|^{2^{n-2}-2}(g-1)x,\ |N|^{2^{n-2}-2}(h-1)y]\\&\quad=|N|^{2^{n-1}-4}\{(g-1)(h^x-1)xy-(h-1)(g^y-1)yx\} \in \Delta_R^{\langle n\rangle}(G).\end{aligned}$$

Thus $\sum_{g\in N}|N|^{2^{n-1}-4}\{(g-1)(h^x-1)xy-(h-1)(g^y-1)yx\}$ also belongs to $\Delta_R^{\langle n\rangle}(G)$. Since $L \leqq N_G(K)$ and $N=(K, L)$, we have $L \leqq N_G(N)$. Thus $g^y \in N$, and we obtain

$$\begin{aligned}&\sum_{g\in N}|N|^{2^{n-1}-4}\{(g-1)(h^x-1)xy-(h-1)(g^y-1)yx\}\\&\quad=|N|^{2^{n-1}-4}\{((\sum_{g\in N} g)-|N|)(h^x-1)xy-(h-1)((\sum_{g\in N} g)-|N|)yx\}.\end{aligned}$$

Here, $(\sum_{g\in N} g)(h^x-1)=(h-1)(\sum_{g\in N} g)=0$ because $h^x, h \in N$. Hence we have

$$|N|^{2^{n-1}-3}\{(1-h^x)xy-(1-h)yx\} \in \Delta_R^{\langle n\rangle}(G).$$

And therefore

$$\begin{aligned}&\sum_{h\in N}|N|^{2^{n-1}-3}\{(1-h^x)xy-(1-h)yx\}\\&\quad=|N|^{2^{n-1}-3}(|N|-\sum_{h\in N} h)(xyx^{-1}y^{-1}-1)yx \in \Delta_R^{\langle n\rangle}(G).\end{aligned}$$

Here, $(\sum_{h\in N} h)(xyx^{-1}y^{-1}-1)=0$ since $xyx^{-1}y^{-1} \in (K, L)=N$. Thus we have

$$|N|^{2^{n-1}-2}(xyx^{-1}y^{-1}-1)yx \in \Delta_R^{\langle n\rangle}(G).$$

Hence $|N|^{2^{n-1}-2}(xyx^{-1}y^{-1}-1) \in \Delta_R^{\langle n\rangle}(G)$ because $\Delta_R^{\langle n\rangle}(G)$ is an ideal. Thus our lemma is proved.

As a special case, we state the following

Corollary. *Let G be a finite group. Then we have*

$$|G'|^{2^{n-1}-2}(x-1) \in \Delta_R^{\langle n\rangle}(G) \qquad \textit{for any } x \in G' \textit{ and } n \geqq 2.$$

Proof. Apply Lemma with $K=L=G$.

3. Proof of the theorem. Now we prove our theorem. First we show the if part. By [3], $\Delta_Z(G')$ is residually nilpotent. On the other hand, as is shown in [1], we have

$$\bigcap_n \Delta_Z^{\langle n\rangle}(G) \subseteqq (\bigcap_n (\Delta_Z(G'))^{2^n})ZG.$$

Thus $\bigcap_n \Delta_Z^{\langle n\rangle}(G)=0$.

Next we consider the converse.

Case 1. There exists a prime p such that $(P, N_G(P)) \neq 1$ where P is a Sylow p-subgroup.

We claim the following.

$(**)$ If $x-1 \in \bigcap_{n,m} \Delta_{Z/p^mZ}^{\langle n\rangle}(G)$ for $x \in G$, we have $x=1$.

Assume that $x-1 \in \bigcap_{n,m} \Delta_{Z/p^mZ}^{\langle n\rangle}(G)$. In other words, $x-1 \in \Delta_Z^{\langle n\rangle}(G)+p^m ZG$ for any n, m. Now pick $1 \neq g \in (P, N_G(P))=N$ and apply Lemma with $K=P$ and $L=N_G(P)$. Then we have $|N|^{2^{n-1}-2}(g-1) \in \Delta_Z^{\langle n\rangle}(G)$. Noting that $|N|$ is a p-power, we have $p^m(g-1) \in \Delta_Z^{\langle n\rangle}(G)$ for large m. Therefore we have $(x-1)(g-1) \in \Delta_Z^{\langle n\rangle}(G)$ for any n, and $(x-1)(g-1) \in \bigcap_n \Delta_Z^{\langle n\rangle}(G)=0$. This implies $x=1$ because $g\neq 1$, and (**) is shown.

Now let q be any prime distinct from p and Q be a Sylow q-subgroup of G (if there are no such primes, G is a p-group and we have nothing to show). Let x be an element in Q'. Then by Corollary and the fact that q is a unit in Z/p^mZ, we have $x-1\in\bigcap_{n,m}\Delta^{\langle n\rangle}_{Z/p^mZ}(Q)\subseteqq\bigcap_{n,m}\Delta^{\langle n\rangle}_{Z/p^mZ}(G)$. Thus we get $x=1$ by (**), and therefore Q is abelian. Hence any two elements of Q which are conjugate in G are conjugate in $N_G(Q)$ by the lemma of Burnside (cf. [2]). Combining the focal subgroup lemma (cf. [2]) with this fact, we have

$$Q\cap G'=\langle x^{-1}y \mid x, y\in Q,\ x\underset{N_G(Q)}{\frown} y\rangle=(Q, N_G(Q)).$$

Now again apply Lemma with $K=Q$ and $L=N_G(Q)$. Then we obtain

$$|Q\cap G'|^{2^{n-1}-2}(x-1)\in\Delta^{\langle n\rangle}_{Z}(G)\qquad\text{for } n\geqq 2 \text{ if } x\in Q\cap G'.$$

This, however, implies that $x-1\in\bigcap_{n,m}\Delta^{\langle n\rangle}_{Z/p^mZ}(G)$ because $q\neq p$. It follows that $x=1$ by (**), and we have $Q\cap G'=1$. Since q is any prime distinct from p, G' is a p-group.

Case 2. For any prime p dividing $|G|$, $(P, N_G(P))=1$ where P is a Sylow p-subgroup.

Since $N_G(P)=C_G(P)$, G is a p-nilpotent group by the theorem of Burnside (cf. [2]) and P is abelian for any p. Therefore G is nilpotent, and in addition, G is abelian. This completes the proof of our theorem.

Remark. We can use the above lemma in a similar way as in the proof of the theorem to simplify the proofs of results in [3], [4], [5] concerning RG with (residually) Lie nilpotent or Lie solvable augmentation ideals when G is a finite group.

References

[1] A. K. Bhandari: Some remarks on the unit groups of integral group rings. Arch. Math., **44**, 319–322 (1985).
[2] D. Gorenstein: Finite Groups. Harper and Row, New York, Evanston and London (1968).
[3] A. I. Lichtman: The residual nilpotence of the augmentation ideals and the residual nilpotence of some classes of groups. Israel J. Math., **26**, 276–293 (1977).
[4] I. Musson and A. Weiss: Integral group rings with residually nilpotent unit groups. Arch. Math., **38**, 514–530 (1982).
[5] M. M. Parmenter, I. B. S. Passi and S. K. Sehgal: Polynomial ideals in group rings. Canad. J. Math., **25**, 1174–1182 (1973).
[6] I. B. S. Passi: Group rings and their augmentation ideals. Lect. Notes in Math. no. 715, Springer-Verlag, Berlin-Heidelberg-New York (1979).
[7] S. K. Sehgal: Topics in Group Rings. Marcel Dekker, New York and Basel (1978).