59. *On Semi-Idempotents in Rings*

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(Communicated by Shokichi IYANAGA, M. J. A., June 10, 1986)

An element $\alpha \neq 0$ of a ring R is called *semi-idempotent* if and only if α is not in the proper two sided ideal generated by $\alpha^2-\alpha$,

$$\text{i.e., } \alpha \notin R(\alpha^2-\alpha)R \quad \text{or} \quad R(\alpha^2-\alpha)R=R.$$

0 is also counted among semi-idempotents. Obviously idempotents and units of R are semi-idempotents.

In [1] W. B. Vasantha proved certain results about semi-idempotents in group rings. We give a generalization of one of the results in the paper and use it to give a characterization of local rings.

Throughout the rest of this paper, R will denote a ring with identity. $\mathrm{Rad}\, R$ denotes the Jacobson radical of R.

Theorem A. *If $\alpha \in R$, then either α is semi-idempotent or $R(1-\alpha)R=R$.*

Proof. If α is not semi-idempotent, we have

$$\alpha \in R(\alpha^2-\alpha)R \subseteq R(1-\alpha)R.$$

Also $(1-\alpha) \in R(1-\alpha)R$. So we have $R(1-\alpha)R=R$.

Remark. This was proved in [1] for the case $R=KG$, K a field, G abelian.

Lemma 1. *Non-zero elements of* $\mathrm{Rad}\, R$ *are not semi-idempotent.*

Proof. Let α be a non-zero element of $\mathrm{Rad}\, R$. As $(1-\alpha)$ is invertible, $R(\alpha^2-\alpha)R=R\alpha R(\subseteq \mathrm{Rad}\, R)$ is therefore a proper ideal containing α. Hence α is not semi-idempotent.

Theorem B. *The following are equivalent for a ring R.*

(1) *$(R/\mathrm{Rad}\, R)$ is a division ring.*

(2) *The only semi-idempotents of R are units and zero.*

Proof. Suppose that the only semi-idempotents of R are units and zero. Consider

$I=\{\alpha \in R \mid R\alpha R \neq R\}$. If we show that I is closed under addition, then I will be a two sided ideal.

Let $\alpha, \beta \in I$. If $\alpha+\beta \notin I$, we have $R(\alpha+\beta)R=R$, i.e., there exists elements a, b, $a \in R\alpha R$, $b \in R\beta R$, such that $a+b=1$.

Neither a nor b can be zero. But as $a \in R\alpha R$, $RaR \neq R$, a is not semi-idempotent by hypothesis. Hence $R(1-a)R=R$ by Theorem A. That is $RbR=R$, which contradicts the fact that $b \in I$.

Hence I is closed under addition. It is easily seen that I is the unique maximal two sided ideal. Now we claim that it is actually a unique maximal left ideal. If α is any non-zero element such that $R\alpha \neq R$, then α is not invertible. So by hypothesis α is not semi-idempotent, i.e., $\alpha \in R(\alpha^2-\alpha)R \neq R$.

From $R\alpha R \subseteq R(\alpha^2-\alpha)R \neq R$ follows now $\alpha \in I$.

Thus we get that any α such that $R\alpha \neq R$ is contained in I. Thus I is the unique maximal left ideal of R. Hence $I = \mathrm{Rad}\, R$ and $(R/\mathrm{Rad}\, R)$ is a division ring.

Conversely, let $(R/\mathrm{Rad}\, R)$ be a division ring. Let α be a non-zero semi-idempotent of R. Then $\alpha \notin \mathrm{Rad}\, R$ by Lemma 1. Hence by assumption α is a unit modulo $\mathrm{Rad}\, R$. This implies, as is well known, that α is a unit.

Remark. If R is commutative, this shows that if R has a unique maximal (two sided) ideal, then the only semi-idempotents of R are units and zero. This need not be true if R is non-commutative. For example in the matrix rings over fields there is a unique maximal two sided ideal. But every element is semi-idempotent.

In [1] the following conjecture was made.

"Let K be a field and $R = KG$ the group ring over any group G. If $\alpha - 1$ is not a unit in R, then α is semi-idempotent". Now we exhibit an example to show that this conjecture is not true.

Let $R = KG$ with $K = Z_2$ and $G = S_3$, where

$$S_3 = \{1, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$$

where σ is the cycle $(1, 2, 3)$ and $\tau = (1, 2)$. Then for $\alpha = 1 + \sigma + \sigma^2 + \tau \in R$, it can be verified that $R\alpha = \alpha R$. Also $\alpha^2 \neq 1$, $\alpha^4 = \alpha^2$ so that α is not a unit in R. Now α is not semi-idempotent since $\alpha^2 - \alpha = 1 + \tau$ and $\alpha = (\alpha^2-\alpha) + \sigma(\alpha^2-\alpha) + (\alpha^2-\alpha)\sigma^2 \in R(\alpha^2-\alpha)R$ also $R(\alpha^2-\alpha)R \subseteq R\alpha R \neq R$.

Therefore α is not semi-idempotent. But $\alpha - 1 = \sigma + \sigma^2 + \tau$ is not a unit in R, since $(\alpha-1)^2 \neq 1$ and $(\alpha-1)^4 = (\alpha-1)^2$.

Acknowledgement. The authors thank the referee for some helpful comments.

Reference

[1] W. B. Vasantha: On semi-idempotents in group rings. Proc. Japan Acad., **61A**, 107–108 (1985).