56. *Discrepancy with respect to Weighted Means of Some Sequences*

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1. It is well-known that for irrational α's of small type the sequences $(n\alpha)$, $n=1, 2, \cdots$, have uniformly low discrepancies [1 : pp. 121–126]. In this note we shall show the connection between the type of α and the P-discrepancy of the sequence $(a_n\alpha)$, $n=1, 2, \cdots$, where (a_n) is a non-decreasing sequence of integers and the P-discrepancy is a generalized notion of discrepancy. Furthermore, we shall give a quantitative form of Theorem 8 of Tsuji [4] with respect to weighted uniform distribution mod 1. This result contains Theorem 4.1 of Niederreiter [3], Satz 3 of Niederreiter and Tichy [2] and Satz 3 of Tichy [5] as special cases.

2. **Definition 1.** Let $P=(p_n)$, $n=1, 2, \cdots$, be a sequence of non-negative real numbers with $p_1>0$. For $N\geqq 1$, put $s_N=p_1+p_2+\cdots+p_N$. Given a sequence $\omega=(x_n)$, $n=1, 2, \cdots$, of real numbers and a positive integer N, the P-discrepancy (mod 1) of the first N terms of ω is defined by

$$D_N(P\,;\,\omega)=\sup_I\left|(1/s_N)\sum_{n=1}^{N} p_n c_I(\{x_n\})-|I|\right|,$$

where the supremum is taken over all intervals I in $[0, 1)$, c_I is the characteristic function of I, $\{x_n\}$ is the fractional part of x_n, and $|I|$ is the length of I.

Definition 2. An irrational number α is said to be of constant type if there exists a constant $C>0$ such that for all integers $q>0$, $q\|q\alpha\|\geqq C$ holds, where $\|t\|=\min_{n\in \boldsymbol{Z}}|t-n|$ for $t\in \boldsymbol{R}$.

Definition 3. Let η be a positive real number or infinity. An irrational number α is said to be of type η if η is the supremum of all γ for which $\underline{\lim}_{q\to\infty}\, q^{\gamma}\|q\alpha\|=0$, where q runs through positive integers.

3. **Results.** Let $p(t)\in C^1[1, \infty)$ be a positive, non-increasing function. We put $p_n=p(n)$ for $n=1, 2, \cdots$. We assume throughout that $\lim_{N\to\infty} s_N=\infty$. Putting $s(t)=\int_1^t p(u)du$ for $t\geqq 1$, the partial sum s_N is asymptotically equal to $s(N)$ as $N\to\infty$.

Theorem 1. *Let $g(t)\in C^2[1, \infty)$ be a positive, strictly increasing function satisfying the following conditions:*

(1) $g(t)\to\infty$ *as* $t\to\infty$,

(2) $g'(t)\to$ *constant* <1 *monotonically as* $t\to\infty$,

(3) $g'(t)/p(t)$ *is monotone for* $t\geqq 1$.

Then for $P=(p(n))$ and $\omega=([g(n)]\alpha)$ with α irrational, there exists an absolute constant C such that

$$D_N(P\,;\,\omega)\leqq C\Big(\frac{1}{m}+\Big(\frac{p(N)}{s(N)g'(N)}+\frac{1}{s(N)}\int_1^N p(t)g'(t)dt\Big)\sum_{h=1}^{m}\frac{1}{h\|h\alpha\|}\Big)$$

for any positive integer m.

Corollary 1. *Let* α *be an irrational number of finite type* η *and let* $g(t)$ *satisfy the same conditions as in Theorem* 1 *but* (3). *Then for every* $\varepsilon>0$, *we have for* $P=(g'(n))$ *and* $\omega=([g(n)]\alpha)$

$$D_N(P\,;\,\omega)\ll\Big(\frac{1}{g(N)}\int_1^N (g'(t))^2dt\Big)^{(1/\eta)-\varepsilon}.$$

Corollary 2. *Let* α *be an irrational number of constant type and let* $g(t)$ *satisfy the same conditions as in Corollary* 1. *Then for* $P=(g'(n))$ *and* $\omega=([g(n)]\alpha)$, *we have*

$$G(N)D_N(P\,;\,\omega)\ll\log^2 G(N),$$

where $G(N)=g(N)\Big/\int_1^N (g'(t))^2dt$.

Theorem 2. *Let* $g(t)$ *satisfy the same conditions as in Theorem* 1. *Then for* $P=(p(n))$ *and* $\omega=(g(n))$, *we have*

$$D_N(P\,;\,\omega)\ll\frac{1}{s(N)}\int_1^N p(t)g'(t)dt+\frac{p(N)}{s(N)g'(N)}.$$

Corollary 3. *Let* $\omega=(\alpha n^{\delta}\log^{\tau} n)$ *with* $\alpha>0$, $0\leqq\delta<1$ *and* τ *such that* $\lim_{n\to\infty} n^{\delta}\log^{\tau} n=\infty$. *Then for* $P=(1/n)$, *we have*

$$D_N(P\,;\,\omega)\ll 1/\log N.$$

4. To prove Theorem 1, we need a well-known theorem.

Lemma (Erdös-Turán [1, p. 114]). *There exists an absolute constant* C *such that*

$$D_N(P\,;\,\omega)\leqq C\Big((1/m)+\sum_{h=1}^{m}(1/h)\Big|(1/s_N)\sum_{n=1}^{N}p_n e^{2\pi i h x_n}\Big|\Big)$$

for any sequence $\omega=(x_n)$ *of real numbers and any positive integer* m.

Proof of Theorem 1. Since $g(t)$ is strictly increasing for $t\geqq 1$ and $g(t)\to\infty(t\to\infty)$, $g(t)$ has an inverse function $f(t)$, $t\geqq 1$. Let m_j be the smallest integer $\geqq f(j)$. For any integer $N\geqq 1$, there exists an integer $k\geqq 1$ such that $N=m_k+r$ with $0\leqq r<m_{k+1}-m_k$. For any positive integer h, we have

$$\sum_{n=1}^{N}p(n)e(h[g(n)]\alpha)=\sum_{n=1}^{m_k-1}p(n)e(h[g(n)]\alpha)+\sum_{n=m_k}^{m_k+r}p(n)e(h[g(n)]\alpha)$$
$$=I_k+R_k,\quad \text{say,}$$

where $e(x)=e^{2\pi i x}$ for real x. Since $f(j+1)-f(j)\geqq 1/g'(f(j))$, by condition (2) there exists an integer j_0 such that $m_{j+1}-m_j\geqq 1$ for $j\geqq j_0$. Hence we may assume without loss of generality that $m_{j+1}-m_j\geqq 1$ for $j\geqq 1$. Now we have

$$I_k=\sum_{j=1}^{k-1}\Big(\sum_{n=m_j}^{m_{j+1}-1}p(n)\Big)e(hj\alpha)+O(1)=\sum_{j=1}^{k-1}q_j e(hj\alpha)+O(1),$$

where $q_j=\sum_{n=m_j}^{m_{j+1}-1}p(n)$. By Euler's summation formula, we have

$$q_j=\int_{f(j)}^{f(j+1)}p(t)dt+O(p(f(j)))=q'_j+O(p(f(j))),$$

where

$$q'_j=\int_{f(j)}^{f(j+1)} p(t)\,dt=\int_j^{j+1} \frac{p(f(t))}{g'(f(t))}\,dt.$$

By condition (3), it follows that (q'_j) is monotone for $j\geqq 1$. Hence, we get

$$\begin{aligned}|I_k|&=\left|\sum_{j=1}^{k-2}(q_j-q_{j+1})\sum_{m=1}^{j}e(hm\alpha)+q_{k-1}\sum_{m-1}^{k-1}e(hm\alpha)+O(1)\right|\\
&\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(\sum_{j=1}^{k-2}|q_j-q_{j+1}|+|q_{k-1}|+O(1)\right)\\
&\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(\sum_{j=1}^{k-2}|q'_{j+1}-q'_j|+|q'_{k-1}|+O\Big(\sum_{j=1}^{k-1}p(f(j))\Big)+O(1)\right)\\
&\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(2q'_{k-1}+O\Big(\sum_{j=1}^{k-1}p(f(j))+O(1)\Big)\right)\\
&\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(2\frac{p(f(k))}{g'(f(k))}+\int_1^{(g)N}p(f(t))\,dt+O(1)\right)\\
&\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(2\frac{p(N)}{g'(N)}+\int_1^N p(t)g'(t)dt+O(1)\right).\end{aligned}$$

Using also Euler's summation formula, by condition (3), we have

$$|R_k|\leqq\frac{p(N)}{g'(N)}+O(1). \tag{5}$$

From (4) and (5) we arrive at

$$\left|\sum_{n=1}^{N}p(n)e(h[g(n)]\alpha)\right|\leqq\frac{1}{|\sin(\pi h\alpha)|}\left(2\frac{p(N)}{g'(N)}+\int_1^N p(t)g'(t)dt+O(1)\right)+\frac{p(N)}{g'(N)}+O(1).$$

Since $1/|\sin(\pi h\alpha)|\leqq 1/2\|h\alpha\|$ for $h\geqq 1$, by Lemma we get the desired inequality. Q.E.D.

5. *Proof of Corollary* 1. Let $\varepsilon>0$ be fixed. It is known that

$$\sum_{h=1}^{m}\frac{1}{h\|h\alpha\|}=O(m^{\eta-1+\varepsilon}),\qquad \text{(see [1], p. 123).}$$

Combining this with Theorem 1, we obtain

$$D_N(P\,;\,\omega)\ll\frac{1}{m}+\frac{\int_1^N (g'(t))^2dt}{g(N)}m^{\eta-1+\varepsilon}.$$

If we choose $m=\left[\left(g(N)\Big/\int_1^N (g'(t))^2\,dt\right)^{1/\eta}\right]$, then we get the desired result.

Q.E.D.

Proof of Corollary 2. It is known that

$$\sum_{h=1}^{m}\frac{1}{h\|h\alpha\|}=O(\log^2 m),\qquad \text{(see [1], p. 124).}$$

Applying Theorem 1, we obtain

$$D_N(P\,;\,\omega)=O\left(\frac{1}{m}+\frac{\log^2 m}{G(N)}\right).$$

If we choose $m=[G(N)]$, we get the desired result. Q.E.D.

6. Applying Euler's summation formula, Theorem 2 follows by the same argument as in [2].

If in Corollary 1 we assume that

$$\int_1^N (g'(t))^2 dt = O(1), \quad \text{then we get } D_N(P\,;\,\omega) \ll (g(N))^{-(1/\eta)+\varepsilon}.$$

This estimate is sharp in the sense that under the same assumptions as in Corollary 1, for every $\varepsilon > 0$, we have $D_N(P\,;\,\omega) = \Omega(g(N)^{-(1/\eta)-\varepsilon})$. By the same reasoning as in the proof of Theorem 3.3 in [1], this Ω-result can be proved.

References

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