54. *Zeta Functions of Analytic Rings via Euler Products*

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

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We try to construct zeta functions of analytic rings by means of Euler products following the formulation of [8] [9]. Here we treat C^* algebras. Detailed studies containing general cases will appear elsewhere.

Let $X: \boldsymbol{R} \to \mathrm{Aut}(A)$ be a C^* dynamical system given by $t \mapsto X_t$ where A is a C^* algebra and $\boldsymbol{R}$ is the additive group of real numbers. Let $\mathrm{FSpec}(A)$ denote the factor spectrum of A, which is the space of equivalence classes of factor representations of A (Pedersen [12, §4.8]). We have naturally a group action $\bar{X}: \boldsymbol{R} \times \mathrm{FSpec}(A) \to \mathrm{FSpec}(A)$. We take up the orbit space $\mathrm{Orb}(\bar{X})$, and we define a subset $P(X)$ of $\mathrm{Orb}(\bar{X})$ and a function $N: P(X) \to \boldsymbol{R}$ such that $N(p)>1$ for all $p \in P(X)$. Then we make the zeta function

$$\zeta^{an}(s, X) = \prod_{p \in P(X)} (1-N(p)^{-s})^{-1}$$

where s is a variable complex number. We consider $\zeta^{an}(s, X)$ as the zeta function $\zeta^{an}(s, C^*(X))$ of the C^* crossed product $C^*(X) = \boldsymbol{R} \ltimes_X A = \bar{A}$ also. The set $P(X) = P_{\mathrm{I}}(X) \cup P_{\mathrm{III}}(X)$ is defined as follows. We say that an orbit $p \in \mathrm{Orb}(\bar{X})$ belongs to $P_{\mathrm{I}}(X)$ if and only if p contains a type I representation (then, p consists of type I representations) and the stabilizer (the isotropy subgroup) $\boldsymbol{R}_p$ is equal to $\boldsymbol{Z} \cdot l(p)$ with $0 < l(p) < \infty$ where $\boldsymbol{Z}$ denotes the integers; in this case we put $N(p) = e^{l(p)}$. Next we denote by $P_{\mathrm{III}}(X)$ the set of fixed points of $\bar{X}$ of type III_λ $(0<\lambda<1)$ satisfying the KMS condition ([2, §5.3] and [12, §8.12]; these points are considered to be "pure phases"); for each $p \in P_{\mathrm{III}}(X)$ we put $N(p) = \lambda(p)^{-1}$ if p is of type $\mathrm{III}_{\lambda(p)}$. (This formulation of $P(X)$ will be seen to be natural by looking the type of the localization $C^*(X)_p$ of $C^*(X)$ at p.) We expect the existence of the Euler datum $E(X) = (P(X), \pi_1(X), \alpha)$ in the sense of [8] [9] which gives L-functions, where $\pi_1(X)$ denotes the conjectural fundamental group of X (or, of $C^*(X)$).

We note two kinds of examples.

Example 1. Let M be a compact space, and let $C(M)$ be the C^* algebra of complex continuous functions on M. Let $X: \boldsymbol{R} \to \mathrm{Aut}(C(M))$ be a C^* dynamical system coming from a group action $\bar{X}: \boldsymbol{R} \times M \to M$. Then $\zeta^{an}(s, X)$ coincides with the zeta function $\zeta(s, \bar{X})$ originally studied by Selberg [15] since $P(X) = P_{\mathrm{I}}(X)$ is consisting of periodic orbits of $\bar{X}$ in $M = \mathrm{Max}(C(M)) = \mathrm{FSpec}(C(M))$. We notice that $P(X)$ can be identified with a quotient space $\mathrm{Prim}(C^*(X))/\sim$ of the primitive ideal space (with respect to the quasi-orbit-equivalence) by the Effros-Hahn conjecture proved by

Gootman-Rosenberg [7]. The simplest case is the following: let $\bar{X}$ be the natural action of $\boldsymbol{R}$ on $M=\boldsymbol{R}/\boldsymbol{Z}$ ($\cong S^1$) then $\zeta^{an}(s, X)=(1-e^{-s})^{-1}$. The Selberg zeta function $\zeta^{an}(s, X)$ has good properties at least when $\bar{X}$ is an Anosov flow on a compact Riemannian manifold M; see Selberg [15], Smale [16], Ruelle [13], and Sunada [17]. We refer to [10, Theorem S] for the functional equation of $\zeta^{an}(s, X)$ when $\bar{X}$ is the geodesic flow on a compact Riemann surface M of genus $g\geqq 2$.

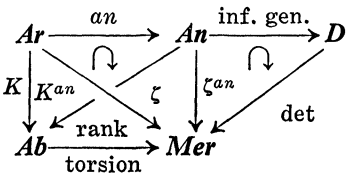
Example 2. Let O_n be the Cuntz algebra studied by Cuntz [4] [5], for $n=2, 3, \cdots$. Let $X: \boldsymbol{R}\to\boldsymbol{R}/2\pi\boldsymbol{Z}\to\mathrm{Aut}(O_n)$ be the gauge action as in Olesen-Pedersen [11] (cf. Bratteli-Robinson [2, 5.3.27] and Evans [6]). Then $\zeta^{an}(s, X)=(1-n^{-s})^{-1}$. In particular $\zeta^{an}(-1, X)=-1/(n-1)=-\sharp K_0(C^*(X))/\sharp K_1(C^*(X))$ since $K_1(C^*(X))\cong K_0(O_n)\cong\boldsymbol{Z}/(n-1)$ and $K_0(C^*(X))\cong K_1(O_n)=0$ by Cuntz [5] and Connes [3]. This interpretation is analogous to the arithmetic case due to Quillen-Lichtenbaum-Beilinson (see [1]).

Some examples containing above ones suggest the following: (1) $\zeta^{an}(s, X)$ would be expressed by a suitable determinant (analytically), and (2) special values of $\zeta^{an}(s, X)$ (for $s\in\boldsymbol{Z}$) would be expressed by (higher) K-groups of $C^*(X)$. Let $D=\lim_{t\to 0}(X_t-1)/t$ be the infinitesimal generator of X assuming the existence in a suitable sense, which will be a skew-hermitian unbounded derivation of A. (When we start from an unbounded derivation D, we put $X_t=\exp(tD)$ and define $\zeta^{an}(s, D)=\zeta^{an}(s, X)$.) We roughly expect that $\zeta^{an}(s, X)=\det(1-D^{-1}s)^{-1}$, which is considered to be a *trace formula* for D^{-1}. For example, if X is the natural action of $\boldsymbol{R}$ on $A=C(\boldsymbol{R}/\boldsymbol{Z})$ noted in Example 1, then D is the usual differential operator d/dx for the variable x on $\boldsymbol{R}/\boldsymbol{Z}$ and the above equality holds (essentially). In general we need to interpret the determinant as a graded determinant of Euler-Poincaré type:

$$\det(1-D^{-1}s)^{-1}=\prod_{m\geqq 0}\det(1-\bar{D}_m^{-1}(s-a_m))^{(-1)^{m+1}}$$

where $\bar{D}_m$ acts on the m-forms $\Omega^m(A)$. Concerning (2) the fact in Example 2 will indicate sufficiently the form (and we add that there is a corresponding expression in Example 1 using K-groups of $C^*(X)$ containing regulators such as analytic torsions especially when X is Anosov or induced from a discrete dynamical system).

We describe schematically the conjectural situation as follows (as noted briefly in [8-I, Remark 1]):



Here ***Ar*** denotes the category of arithmetic rings (finitely generated commutative $\boldsymbol{Z}$-algebras); ***An*** the category of "analytic rings" which is here identified with the category of crossed products $C^*(X)$ attached to C^* dynamical systems $X: \boldsymbol{R} \to \mathrm{Aut}(A)$; $an: \boldsymbol{Ar} \to \boldsymbol{An}$ a zeta (and K) preserving functor: $\zeta(s, A) = \zeta^{an}(s, A^{an})$ for each arithmetic ring A, where $\zeta(s, A) = \prod_{p \in \mathrm{Max}(A)} (1 - N(p)^{-s})^{-1}$ with $N(p) = \sharp(A/p)$, the cardinality of the finite field A/p, and p runs over the maximal ideal space $\mathrm{Max}(A)$; $\boldsymbol{Ab}$ the category of abelian groups; $\boldsymbol{D}$ a suitable category of "differential (Dirac) operators", and $\boldsymbol{Mer}$ the (discrete) category of meromorphic functions on $\boldsymbol{C}$.

From the view point above, the gauge crossed products of Cuntz algebras are considered to be analytic analogues of finite fields: $\boldsymbol{F}_q^{an} = \bar{O}_q = \boldsymbol{R} \ltimes O_q$ for each finite field $\boldsymbol{F}_q$. (Symbolically speaking $\boldsymbol{F}_q^{an}$ is a generalized Fermion algebra "$\boldsymbol{F}_q$".) Note that $\zeta(s, \boldsymbol{F}_q) = (1 - q^{-s})^{-1}$, $K_0(\boldsymbol{F}_q) = 0$ (the reduced K_0 group, i.e. the projective class group) and $K_1(\boldsymbol{F}_q) \cong \boldsymbol{Z}/(q-1)$. Our formulation is compatible with making direct sums of rings. For example:

Theorem. *Let* $A = O_{n_1} \oplus \cdots \oplus O_{n_r}$ *for* $n_1, \cdots, n_r \geqq 2$ *and let* $X: \boldsymbol{R} \to \mathrm{Aut}(A)$ *be the gauge action. Then*

$$\zeta^{an}(s, X) = (1 - n_1^{-s})^{-1} \cdots (1 - n_r^{-s})^{-1}$$

and

$$\zeta^{an}(-1, X) = (-1)^r \cdot \sharp K_0(C^*(X)) / \sharp K_1(C^*(X)).$$

Remark 1. For a C^* dynamical system $X: \boldsymbol{R}^n \to \mathrm{Aut}(A)$ similar formulation is possible with $P(X) = P_{\mathrm{I}}(X) \cup P_{\mathrm{III}}(X)$ where $P_{\mathrm{I}}(X)$ is the set of "periodic orbits" (orbits satisfying $\boldsymbol{R}_p^n \cong \boldsymbol{Z}^n$ with $l(p) = \mathrm{vol}(\boldsymbol{R}^n / \boldsymbol{R}_p^n)$) of type I and $P_{\mathrm{III}}(X)$ is the set of $\boldsymbol{R}^n$-KMS representations of type III.

Remark 2. The following $F(A)$ (cf. Cuntz [4] and Evans [6]) is similar to A^{an} for $A \in \boldsymbol{Ar}$ in some sense:

$$F(A) = \boldsymbol{R} \underset{\text{gauge}}{\ltimes} (\boldsymbol{Z} \underset{\text{shift}}{\ltimes} F_0(A))$$

where

$$F_0(A) = \Big(\coprod_{-\infty}^{\infty} E(A) \Big) \ltimes C_0 \Big(\coprod_{-\infty}^{-1} E(A) \times \prod_{0}^{\infty} E(A) \Big),$$

$E(A)$ being the pro-finite completion (or the "Euler product ring") of A, and the gauge action is the dual of the shift.

Remark 3. The zeta function $\zeta(s, U)$ of the universe U (in the superstring formulation) would be identified with $\zeta^{an}(s, X)$ for the "canonical" dynamical system X on the "superstring algebra" A, which will be a super-version of a C^* algebra (or a super-version of a Jordan $*$ algebra such as $C(S^1, H_3(\boldsymbol{O}))$; note that all exceptional groups—for example E_8 appearing in the superstring theory—are constructed from the 27 dimensional exceptional Jordan algebra $H_3(\boldsymbol{O})$ by the method of Tits and the (bounded) derivation algebra $\mathrm{Der}\, C(S^1, H_3(\boldsymbol{O})) \cong C(S^1, \mathrm{Der}\, H_3(\boldsymbol{O}))$ is a loop algebra over the 52 dimensional Lie algebra $\mathrm{Der}\, H_3(\boldsymbol{O})$ of type F_4). We refer to Schwarz [14] and Witten [18] for superstring algebras. In super-versions, the time variable is naturally considered to be a 1-form. The *super trace formula* is essential.

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