6. $\bar{\partial}$-Problem on a Family of Weakly Pseudoconvex Manifolds

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1. In the case of weakly pseudoconvex manifolds the $\bar{\partial}$-problem depends not only on boundary conditions, but also on complex structures ([2, 3, 4, 5, 7, 8]). In this paper we investigate the $\bar{\partial}$-problem for the Picard variety $\mathrm{Pic}^0(T^n)$ of a complex n-dimensional torus T^n, which is regarded as a family of weakly pseudoconvex manifolds. For this problem we find a criterion given by the theory of Diophantine approximation. Full details will be published elsewhere.

2. Let $E \in \mathrm{Pic}^0(T^n)$. Then E is a holomorphic line bundle on T^n with Chern class zero. By a result of [1] we find a proper weakly plurisubharmonic C^∞-function $\Phi : E \to [0, \infty)$. Thus, we can regard $\mathrm{Pic}^0(T^n)$ as a family of noncompact weakly pseudoconvex manifolds. Since $\mathrm{Pic}^0(T^n)$ is isomorphic to a complex n-dimensional torus $T^{n*} = C^n/\Lambda$, we can define on $\mathrm{Pic}^0(T^n)$ an invariant distance $d(E, F) := \min\{\|a-b+c\|; E = a+\Lambda, F = b+\Lambda \in C^n/\Lambda, c \in \Lambda\}$, where $\|(z_1, \cdots, z_n)\| := \max |z_i|$.

Theorem. *Let $E \in \mathrm{Pic}^0(T^n)$ and O the structure sheaf of E. Then E must be one of the following types:*

(1) *If $d(1, E^l) = 0$ for some $l \geqq 1$, then $H^p(E, O)$ is an infinite-dimensional Hausdorff space $(1 \leqq p \leqq n)$;*

(2) *If there exists $a > 0$ such that $\exp(-al) \leqq d(1, E^l)$ for any $l \geqq 1$, then $\dim H^p(E, O) = \binom{n}{p}$ $(1 \leqq p \leqq n)$;*

(3) *If $d(1, E^l) \neq 0$ for any $l \geqq 1$ and $\liminf_{l\to\infty} \exp(al) d(1, E^l) = 0$ for any $a > 0$, then $H^p(E, O)$ is not Hausdorff $(1 \leqq p \leqq n)$.*

Further let P_1, P_2 and P_3 be the subsets of $\mathrm{Pic}^0(T^n)$ consisting of the elements of the above types (1), (2) and (3), respectively. Then P_i is nonempty $(i = 1, 2, 3)$, $P_1 \cup P_3$ is of Lebesgue measure zero and

$$\mathrm{Pic}^0(T^n) = P_1 \cup P_2 \cup P_3 \quad \text{(disjoint)}.$$

Remark. In comparison with strongly pseudoconvex manifolds, this theorem shows that a strange phenomenon occurs for weak pseudoconvexity. We have $P_2 \cup P_3 = \{E \in \mathrm{Pic}^0(T^n); H^0(E, O) = C\}$. E contains T^n as its zero section. If $E \in P_2$, then $H^p(E, O) \cong H^p(T^n, O_{T^n})$. This is similar to the case of a strongly pseudoconvex manifold and its exceptional set. But if $E \in P_3$, there exists a great difference from strong pseudoconvexity.

Proof. Let $E \in \mathrm{Pic}^0(T^n)$. We have a bireal-analytic isomorphism T^n

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$\ni z \mapsto t(z)=(t_1(z), \cdots, t_{2n}(z)) \in R^{2n}/Z^{2n}$ and a real analytic map $\zeta : E \ni x \mapsto \zeta(x) \in C$ such that $\zeta | \pi^{-1}(z)$ is a biholomorphic isomorphism $\pi^{-1}(z) \cong C$ for each $z \in T^n$, where π is the projection of E to T^n. Then $i : E \ni x \mapsto (\zeta(x), t \circ \pi(x)) \in C \times R^{2n}/Z^{2n}$ is bireal-analytic in E and holomorphic along each fiber of E. Let U be an open set in E, $H(U)=\{f$; real analytic in U and holomorphic in $\pi^{-1}(z) \cap U$ for any $z \in T^n\}$ and H the sheaf defined by the presheaf $\{H(U)\}$. In [3] we have proved the following lemma. And Prof. Siciak informed the authors of another proof of the lemma, using a result of [6].

Lemma. *For* $\{a^{m,l} \in C;\ m=(m_1, \cdots, m_{2n}) \in Z^{2n},\ l=0, 1, 2, \cdots\}$ *we set, formally* $\phi(\zeta, t) := \sum_{m \in Z^{2n}} \sum_{l=0}^{\infty} a^{m,l} e_m(t) \zeta^l$ *on* $C \times R^{2n}/Z^{2n}$, *where* $e_m(t) = \exp 2\pi\sqrt{-1}(m_1 t_1 + \cdots + m_{2n} t_{2n})$. *Then* $\phi \circ i \in H^0(E, H)$, *if and only if there exists* $\varepsilon > 0$ *such that* $\sup\{|a^{m,l}| \exp(\varepsilon \|m\| + al);\ m \in Z^{2n},\ l=0, 1, 2, \cdots\} < +\infty$ *for any* $a > 0$.

We put $H^{p,q} := H \otimes \pi^* A^{p,q}(T^n)$, where $A^{p,q}(T^n)$ denotes the sheaf of germs of real analytic forms of type (p, q) on T^n. Then we have $H^p(E, O) = \{\phi \in H^0(E, H^{0,p});\ \bar{\partial}\phi = 0\} / \bar{\partial} H^0(E, H^{0,p-1})$ (cf. [3]). For $\phi \in H^0(E, H^{0,p})$ we have the Fourier-Taylor expansion

$$\phi = \frac{1}{p\,!} \sum_{i_1, \cdots, i_p, m \in Z^{2n}} \sum_{l=0}^{\infty} a^m_{i_1 \cdots i_p} e_m(t) \zeta^l \pi^* d\bar{\zeta}_{i_1} \wedge \cdots \wedge \pi^* d\bar{\zeta}_{i_p},$$

where $\{d\zeta_i\}$ denotes a system of global forms of type $(0, 1)$ on T^n. We can write the $\bar{\partial}$-operator by coefficients of Fourier-Taylor expansions of forms and, similarly to an argument of [3], we have a formal solution for the $\bar{\partial}$-problem. To complete the proof, we check, using Lemma, whether the formal solution is convergent or not.

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