53. On Compactifiable Strongly Pseudoconvex Surfaces

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Throughout, analytic surfaces will mean 2-dimensional C-analytic manifolds. Purely 1-dimensional C-analytic spaces will be referred to simply as analytic curves. Furthermore, all compact analytic surfaces are assumed to be *minimal* [7], i.e., free from exceptional curves of the first kind.

1. Structures of compactifiable strongly pseudoconvex surfaces.

Definition 1 [8], [9]. A non compact analytic surface X is said to be *strongly pseudoconvex* if i) X is holomorphically convex and if ii) there exists a compact analytic curve $E\subset X$ such that $T\subseteq E$ for any irreducible compact analytic curve $T\subset X$.

E is called the *exceptional* curve of X. In the special case where $E=\phi$, X is called a *Stein* surface.

Definition 2. Let X be a non compact analytic surface. A compact analytic surface M is said to be a *compactification* of X if there exists a C-analytic subvariety $\Gamma\subset M$ such that X is biholomorphic to $M\backslash\Gamma$. Furthermore, M is said to be an algebraic (or a non algebraic) compactification if M is an algebraic (or a non algebraic) surface. X is called *compactifiable* if it admits some compactification M.

Remark 1. If X is a strongly pseudoconvex or a Stein surface, then one can check that Γ is a compact connected analytic curve [3].

Our main goal here is to investigate the global structures of i) compactifiable Stein surfaces and ii) compactifiable strongly pseudoconvex surfaces which are not Stein (i.e. $E\neq\phi$).

Remark 2. In view of Definition 1, Stein surfaces are merely special cases of strongly pseudoconvex surfaces; so one might wonder why the treatment of those two surfaces has to be dealt with separately. One of our main purposes here is to point out the sharp contrast between those two cases from the view point of compactification. Hence, throughout, strongly pseudoconvex surfaces are meant to be not Stein!

Our investigation is motivated by the following results:

Theorem 1 [3], [11]. *Let M be a compactification of some Stein surface X. Then M is either* i) *an algebraic surface,* ii) $b_1=1$, $b_2=0$, *M admits no non constant meromorphic functions and contains exactly one compact analytic curve or* iii) $b_1=1$, $b_2>0$ *and M contains exactly one compact analytic curve.*

Theorem 2 [9]. *Let M be a compactification of some strongly pseudoconvex surface X. Then M is either* i) *an algebraic surface, or* ii) $b_1=1$, $b_2=n>0$ *and M contains at least two compact connected analytic curves.*

(Here b_1 and b_2 denote the first and the second Betti number for M, respectively.)

2. Existence of compactifications. The alternatives in Theorems 1 and 2 do indeed occur; in fact, one has the following

Example 1. Let $\boldsymbol{H}_\alpha := C^2\backslash\{0\}/G_\alpha$ where G_α is the subgroup of $GL(2, C)$ generated by matrices of the form

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad \text{with } 0<|\alpha|<1.$$

It is known [7] that $b_1(\boldsymbol{H}_\alpha)=1$, $b_2(\boldsymbol{H}_\alpha)=0$ and $\boldsymbol{H}_\alpha$ admits no non constant meromorphic functions and contains exactly one compact analytic curve $\Gamma_\alpha := C^*/\langle\alpha\rangle$. Furthermore, one can check that [3], [11] $\boldsymbol{H}_\alpha\backslash\Gamma_\alpha \cong C^*\times C^* =: X_1$.

Example 2. In 1978, Kato [6] exhibited a compact analytic surface M with $b_1=b_2=1$ and M contains exactly one compact analytic curve Γ, a rational curve with one ordinary double point and $\Gamma^2=0$. Furthermore, one can check that $X_2 := M\backslash\Gamma$ is a Stein surface; in fact X_2 is biholomorphic to an affine C-bundle of degree -1 over some non singular elliptic curve Λ.

Example 3. In 1974, Inoue [4] explicitly exhibited a compact analytic surface M with $b_1=b_2=1$ and M contains exactly two compact analytic curves: a) a rational curve Γ with one ordinary double point and $\Gamma^2=0$, and b) a non singular elliptic curve Λ with $\Lambda^2=-1$.

Furthermore [8] $X_3 := M\backslash\Gamma$ is a strongly pseudoconvex surface admitting Λ as its exceptional set.

Remark 3. The compact surfaces in Example 1 are called *non elliptic Hopf* surfaces [7]. Meanwhile the compact surfaces M in Examples 2 and 3 are called *parabolic Inoue* surfaces [1], [2].

Now, notice that X_1 also does admit $\boldsymbol{P}_1\times\boldsymbol{P}_1$ as its algebraic compactification. On the other hand, X_2 (or X_3) admits a $\boldsymbol{P}_1$-bundle over Λ, i.e., an elliptic ruled surface as its algebraic compactification.

In view of this strange phenomenon, one would like to raise the following

Question 1. Let X be a Stein surface (or a strongly pseudoconvex surface). If X admits a non algebraic compactification, does X always admit an algebraic compactification?

Notice that the converse to Question 1 is false; in fact, there exist Stein and strongly pseudoconvex surfaces of which the compactifications are always algebraic [11].

Furthermore, Question 1 stems from the following

Problem 1. Let X be a compactifiable Stein (or strongly pseudoconvex) surface. Does X always admit an algebraic structure?

In [9], [11], an affirmative answer to Problem 1 as well to Question 1 was given, namely

Theorem 3. *Let X be a Stein (or a strongly pseudoconvex) surface. Then X is compactifiable iff X admits some algebraic structure.*

Now Problem 1 can be sharpened as follows:

Problem 1′. Let X be a compactifiable Stein surface. Does X always admit some affine structure?

We refer to [10], [11] for more detailed discussions on this problem.

3. Uniqueness of compactifications. Naturally the question of uniqueness of compactification comes up to our mind. However in view of results in section 2, our request can be formulated as follows:

Problem 2. Let M_i $(i=1, 2)$ be two algebraic (or non algebraic) compactifications of some Stein surface X. Are M_i birationally (or bimeromorphically) equivalent?

Problem 3. Let M_i $(i=1, 2)$ be two algebraic (or non algebraic) compactifications of some strongly pseudoconvex surface X. Are M_i birationally (or bimeromorphically) equivalent?

As was explained in Remark 2, the answer for Problem 2 is no. Meanwhile Problem 3 admits an affirmative answer!

Counterexample 1. From a remark, due to Serre, [11] $C^* \times C^*$ is biholomorphic (as abstract algebraic group) to an extension of some elliptic curve Λ by C. Hence, one can check that $X := C^* \times C^*$ admits M_1, a $\boldsymbol{P}_1$-bundle over Λ, as an analytic compactification. On the other hand, X also admits $M_2 := \boldsymbol{P}_1 \times \boldsymbol{P}_1$ as another algebraic compactification. Hence X admits two algebraic compactifications which are not birationally equivalent: M_1 a non rational surface and M_2 a rational surface.

Counterexample 2. By using the same notations as in Example 1, let $\boldsymbol{H}_\alpha$ and $\boldsymbol{H}_\beta$ with $\alpha \neq \beta$ be two non elliptic Hopf surfaces containing, respectively, exactly one compact analytic curve Γ_α and Γ_β. Since $\boldsymbol{H}_\alpha \backslash \Gamma_\alpha \cong \boldsymbol{H}_\beta \backslash \Gamma_\beta \cong C^* \times C^* =: X$, hence the latter admits two non algebraic compactifications which are not bimeromorphically equivalent since $\alpha \neq \beta$.

Despite this current trend, compactifications of strongly pseudoconvex surfaces are of different nature. In fact, if a ruled surface M occurs as an algebraic compactification of some strongly pseudoconvex surface X, one can check [10] that M is uniquely determined by the exceptional curve E of X. Furthermore, one has the following

Lemma 4 [10]. *Let M be an algebraic compactification of some strongly pseudoconvex surface X. If M is not ruled, then $\kappa(X)=2$. (Here $\kappa(X)$ denotes the logarithmic Kodaira dimension of X in the sense of Iitaka* [5].)

Now an affirmative answer for the first half of Problem 3 can be stated as follows:

Theorem 5 [10]. *Let M_i $(i=1, 2)$ be two algebraic compactifications of some strongly pseudoconvex surface X. Then M_i are birationally equi-*

valent.

Corollary 6 [11]. *Let M_i $(i=1,2)$ be two algebraic compactifications of some Stein surface X. Let us assume that M_i are not ruled surface. Then M_i are biholomorphically equivalent.*

By comparing this with Counterexample 1, one might ask:

Question 2. Is $C^* \times C^*$ (up to biholomorphism) the only Stein surface admitting non birationally equivalent algebraic compactifications?

On the other hand on the basis of profound study by Enoki [1], [2] on compact surfaces with $b_1=1$, $b_2>0$ with compact curves D such that $D^2=0$, a positive answer for the second half of Problem 3 can be stated as follows:

Theorem 7 [10]. *Let M_i $(i=1,2)$ be two non algebraic compactifications of some strongly pseudoconvex surface X. Then M_i are biholomorphically equivalent.*

Corollary 8 [11]. *$C^* \times C^*$ is (up to biholomorphism) the only Stein surface admitting non algebraic compactifications which are not bimeromorphically equivalent.*

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