52. *Area Integrals for Normal and Yosida Functions*

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1. Introduction. We shall consider necessary and sufficient conditions for a meromorphic function to be normal or Yosida ([1], [6]).

A function f meromorphic in $D=\{|z|<1\}$ is called normal if

$$k(f)=\sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty,$$

where $f^{\#}=|f'|/(1+|f|^2)$ is the spherical derivative. In terms of the non-Euclidean hyperbolic distance:

$$\sigma(z, w)=\tanh^{-1}|\phi_w(z)|,$$

where

$$\phi_w(z)=(z-w)/(1-\overline{w}z), \qquad z, w\in D,$$

the non-Euclidean open disk of center $a\in D$ and radius $\tanh^{-1}\rho$ $(0<\rho\leqq 1)$ is given by

$$\Delta(a, \rho)=\{|\phi_a(z)|<\rho\}.$$

Theorem 1. *Let f be meromorphic in D. Then the following are mutually equivalent.*

(1) *f is normal.*

(2) *For each $A>0$ there exists $\rho\in(0,1)$ such that*

$$\sup_{a\in D}\sup_{z\in\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|<A. \tag{1.1}$$

(3) *There exist ρ and λ in $(0,1)$ such that*

$$\sup_{\lambda<|a|<1}\iint_{\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2\frac{dxdy}{(1-|z|^2)^2}<\infty. \tag{1.2}$$

Here, $(f(z)-f(a))/(1+\overline{f(a)}f(z))=1/f(z)$ if $f(a)=\infty$. We note that $(1-|z|^2)^{-2}dxdy$ is the non-Euclidean area element at $z=x+iy\in D$.

A function f meromorphic in $C=\{|z|<\infty\}$ is called Yosida if

$$l(f)=\sup_{z\in C}f^{\#}(z)<\infty.$$

See [2], [3], [4], and [5]. We next consider the Euclidean disks:

$$U(a, \rho)=\{|z-a|<\rho\}, \quad a\in C, \quad \rho>0.$$

Theorem 2. *Let f be meromorphic in C. Then the following are mutually equivalent.*

(4) *f is Yosida.*

(5) *For each $A>0$ there exists $\rho\in(0,\infty)$ such that*

$$\sup_{a\in C}\sup_{z\in U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|<A. \tag{1.3}$$

(6) *There exist ρ and λ in $(0,\infty)$ such that*

$$\sup_{\lambda<|a|<\infty}\iint_{U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2 dxdy<\infty. \tag{1.4}$$

Let N_0 be the family of meromorphic functions f in D such that

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z)=0,$$

and let Y_0 be the family of meromorphic functions f in $\boldsymbol{C}$ such that

$$\lim_{|z|\to\infty} f^{\#}(z)=0.$$

Obviously each f of N_0 (Y_0) is normal (Yosida). The results similar to Theorems 1 and 2 can be proved for f to be of N_0 and Y_0, and will be proposed in Section 4.

The situation resembles that for holomorphic case, the Bloch-function criteria. This time, $f-f(a)$ instead of $(f-f(a))/(1+\overline{f(a)}f)$ for f holomorphic in D should be considered. However, in this case, we can prove much more, and the details will be published in the other paper.

2. Proof of Theorem 1. The chordal distance $X(z, w)\geqq 0$ of z and w in $\boldsymbol{C}\cup\{\infty\}$ is given by

$$X(z, w)^2=|F_w(z)|^2/[1+|F_w(z)|^2],$$

where

$$F_w(z)=(z-w)/(1+\bar{w}z) \qquad \text{if } w\neq\infty,$$
$$=1/z \qquad \text{if } w=\infty.$$

In the proof of (1)⇒(2) we first note that

$$X(f(z), f(w))\leqq k\sigma(z, w), \quad z, w\in D, \quad k=k(f).$$

Given $A>0$ we can find $\rho\in(0, 1)$ such that

$$k^2\sigma(\rho, 0)^2<A^2/(1+A^2).$$

Then, for each $z\in\Delta(a, \rho)$, with $P=X(f(z), f(a))$, we have

$$|F_{f(a)}\circ f(z)|^2=\frac{P^2}{1-P^2}\leqq\frac{k^2\sigma(\rho, 0)^2}{1-k^2\sigma(\rho, 0)^2}<A^2,$$

whence (2).

Since the proof of (2)⇒(3) is easy, it remains to observe (3)⇒(1). By the square integral condition, the meromorphic function $F_{f(a)}\circ f$ has no pole in $\Delta(a, \rho)(\lambda<|a|<1)$. Then, there exists a holomorphic function g in the disk $\{|w|<\rho\}$ such that

$$h(w)\equiv F_{f(a)}\circ f\circ\phi_{-a}(w)=wg(w), \qquad |w|<\rho.$$

Furthermore,

$$|g(0)|=(1-|a|^2)f^{\#}(a). \tag{2.1}$$

Since $\log|g|$ is subharmonic in $\{|w|<\rho\}$, it follows that

$$\log|g(0)|\leqq\frac{1}{\pi\rho^2}\iint_{|w|<\rho}\log|g(w)|\,dudv \quad (w=u+iv)$$
$$=\frac{1}{\pi\rho^2}\iint_{|w|<\rho}\log|h(w)|\,dudv+\frac{1}{\pi\rho^2}\iint_{|w|<\rho}\log\frac{1}{|w|}dudv$$
$$\leqq\frac{1}{2}\log\left[\frac{1}{\pi\rho^2}\iint_{|w|<\rho}|h(w)|^2\,dudv\right]+\log\frac{1}{\rho}+\frac{1}{2},$$

whence

$$(1-|a|^2)f^{\#}(a)\leqq\rho^{-1}e^{1/2}\left[\frac{1}{\pi\rho^2}\iint_{|w|<\rho}|h(w)|^2\,dudv\right]^{1/2}. \tag{2.2}$$

On the other hand, since

$$(1-|z|^2)|\phi_a'(z)|\leqq 1 \qquad (z\in D),$$

it follows on setting $w=\phi_a(z)$ that

$$\text{(2.3)}\qquad \frac{1}{\pi\rho^2}\iint_{|w|<\rho}|h(w)|^2\,dudv$$
$$=\frac{1}{\pi\rho^2}\iint_{\Delta(a,\rho)}|F_{f(a)}\circ f(z)|^2|\phi_a'(z)|^2\,dxdy$$
$$\leqq\frac{1}{\pi\rho^2}\iint_{\Delta(a,\rho)}|F_{f(a)}\circ f(z)|^2\frac{dxdy}{(1-|z|^2)^2}.$$

It now follows from (2.2) and (2.3) that

$$\text{(2.4)}\qquad (1-|a|^2)f^{\#}(a)\leqq\rho^{-1}e^{1/2}\left[\frac{1}{\pi\rho^2}\iint_{\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2\frac{dxdy}{(1-|z|^2)^2}\right]^{1/2},$$

where $\lambda<|a|<1$. Since $f^{\#}$ is continuous in D, it follows that f is normal in D.

3. Proof of Theorem 2. The proof is in spirit the same as that of Theorem 1 and will be only sketched.

For the proof of (4)⇒(5) we note that

$$X(f(z),f(a))\leqq l\rho \qquad \text{for } z\in U(a,\rho),\ l=l(f).$$

Letting $\rho>0$ so small that $l^2\rho^2<A^2/(1+A^2)$, we obtain

$$|F_{f(a)}\circ f(z)|^2\leqq\frac{l^2\rho^2}{1-l^2\rho^2}<A^2, \qquad z\in U(a,\rho).$$

Since (5)⇒(6) is trivial we prove (6)⇒(4). This time we consider

$$F_{f(a)}\circ f(w+a)=wg(w), \qquad |w|<\rho.$$

Then, $|g(0)|=f^{\#}(a)$, and our final estimate corresponding to (2.4) is

$$\text{(3.1)}\qquad f^{\#}(a)\leqq\rho^{-1}e^{1/2}\left[\frac{1}{\pi\rho^2}\iint_{U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2dxdy\right]^{1/2}.$$

4. N_0 and Y_0 functions. It is now an easy exercise to prove the following, in particular, with the aid of (2.4) and (3.1).

Theorem 3. *Let f be meromorphic in D. Then the following are mutually equivalent.*

(7) $f\in N_0$.

(8) *For each* $\rho\in(0,1)$,

$$\lim_{|a|\to1}\sup_{z\in\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|=0.$$

(9) *There exists* $\rho\in(0,1)$ *such that*

$$\lim_{|a|\to1}\iint_{\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2\frac{dxdy}{(1-|z|^2)^2}=0.$$

Theorem 4. *Let f be meromorphic in $\mathbf{C}$. Then the following are mutually equivalent.*

(10) $f\in Y_0$.

(11) *For each* $\rho\in(0,\infty)$,

$$\lim_{|a|\to\infty}\sup_{z\in U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|=0.$$

(12) *There exists* $\rho\in(0,\infty)$ *such that*

$$\lim_{|a|\to\infty}\iint_{U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2 dxdy=0.$$

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