49. The Aitken-Steffensen Formula for Systems of Nonlinear Equations. III

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(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1986)

1. Introduction. Let $x=(x_1, x_2, \cdots, x_n)$ be a vector in R^n and D a region contained in R^n. Let $f_i(x)$ $(1 \leqq i \leqq n)$ be real-valued nonlinear functions defined on D and $f(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ an n-dimensional vector-valued function. Then we shall consider a system of nonlinear equations

(1.1) $$x=f(x),$$

whose solution is $\bar{x}$.

As mentioned in [2], [3] and [4], Henrici [1, p. 116] has considered a formula, which is called the Aitken-Steffensen formula. Now, we have studied the above Aitken-Steffensen formula in [2] and [4], and shown [2, Theorem 2] and [4, Theorem 2]. Moreover, by considering the Steffensen iteration method, we have also shown [3, Theorem 1], which improves the result of [2, Theorem 2].

The purpose of this paper is to show Theorem 1 having a new relation different from [2, Theorem 2], [3, Theorem 1] and [4, Theorem 2].

2. Statement of results. Let $U(\bar{x})=\{x ; \|x-\bar{x}\|<\delta\} \subset D$ be a neighbourhood. Let $\|x\|$ and $\|A\|$ be denoted by

$$\|x\|=\max_{1 \leqq i \leqq n}|x_i| \quad \text{and} \quad \|A\|=\max_{1 \leqq i \leqq n} \sum_{j=1}^{n}|a_{ij}|,$$

where $A=(a_{ij})$ is an $n \times n$ matrix.

Given $x^{(0)} \in R^n$, define $x^{(i)} \in R^n$ $(i=1, 2, \cdots)$ by

(2.1) $$x^{(i+1)}=f(x^{(i)}) \qquad (i=0, 1, 2, \cdots).$$

Put

(2.2) $$d^{(i)}=x^{(i)}-\bar{x} \qquad \text{for } i=0, 1, 2, \cdots,$$

and then define an $n \times n$ matrix D_k by

$$D_k=(d^{(k)}, d^{(k+1)}, \cdots, d^{(k+n-1)}).$$

Throughout this paper, we shall assume the same conditions (A.1)–(A.5) as in [2].

(A.1) $f_i(x)$ $(1 \leqq i \leqq n)$ are two times continuously differentiable on D.

(A.2) There exists a point $\bar{x} \in D$ satisfying (1.1).

(A.3) $\|J(\bar{x})\|<1$, where $J(x)=(\partial f_i(x)/\partial x_j)$ $(1 \leqq i, j \leqq n)$.

(A.4) The vectors $d^{(k)}, d^{(k+1)}, \cdots, d^{(k+n-1)}$, $k=0, 1, 2, \cdots$, are linearly independent.

(A.5) $\inf\{|\det D_k|/\|d^{(k)}\|^n\}>0$.

Then, we shall consider the Aitken-Steffensen formula

(2.3) $$y^{(k)}=x^{(k)}-\Delta X^{(k)}(\Delta^2X^{(k)})^{-1}\Delta x^{(k)} \qquad (k=0,1,2,\cdots),$$
where an n-dimensional vector $\Delta x^{(k)}$, and $n\times n$ matrices $\Delta X^{(k)}$ and $\Delta^2X^{(k)}$ are given by

(2.4) $$\Delta x^{(k)}=x^{(k+1)}-x^{(k)},$$

(2.5) $$\Delta X^{(k)}=(x^{(k+1)}-x^{(k)},\cdots,x^{(k+n)}-x^{(k+n-1)})$$

and

(2.6) $$\Delta^2X^{(k)}=\Delta X^{(k+1)}-\Delta X^{(k)}.$$

In this paper, we shall show the following

Theorem 1. *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *a new relation of the form*
$$\|y^{(k+1)}-\bar{x}\|\leqq M\|y^{(k)}-\bar{x}\|+\varepsilon_k, \qquad \varepsilon_k\to 0\ (k\to\infty)$$
holds with a constant M *satisfying* $\|J(\bar{x})\|<M<1$, *where* ε_k *can be considered as "convergent term".*

Remark 1. It follows from [2, Theorem 1] that $x^{(k)}\to\bar{x}$ as $k\to\infty$, and so, by [2, Theorem 2], $y^{(k)}\to\bar{x}$ as $k\to\infty$.

3. Preliminaries. As mentioned in [2], we have, by (2.1), (2.2) and (A.2),

(3.1) $$d^{(k+1)}=J(\bar{x})d^{(k)}+\xi(x^{(k)}),$$

$\xi(x^{(k)})$ being an n-dimensional vector, and by (A.1),

(3.2) $$\|\xi(x^{(k)})\|\leqq L_1\|d^{(k)}\|^2 \qquad \text{for } x^{(k)}\in U(\bar{x}),$$

a constant L_1 being suitably chosen.

Define an $n\times n$ matrix $Y(x^{(k)},\cdots,x^{(k+n)})$ by
$$Y(x^{(k)},\cdots,x^{(k+n)})=(\xi(x^{(k+1)})-\xi(x^{(k)}),\cdots,\xi(x^{(k+n)})-\xi(x^{(k+n-1)})).$$
Then, we have shown in [2] that there exist constants L_2 and L_3 such that the inequalities

(3.3) $$\|Y(x^{(k)},\cdots,x^{(k+n)})\|\leqq L_2\|d^{(k)}\|^2,$$

(3.4) $$\|\Delta X^{(k)}\|\leqq L_3\|d^{(k)}\|$$

hold for $x^{(k)}\in U(\bar{x})$.

For the proof of Theorem 1, we need the following two lemmas. Lemma 1 follows from [2, Theorem 1].

Lemma 1. *Under conditions* (A.1)–(A.3), *we have*

(3.5) $$\|x^{(k+1)}-\bar{x}\|\leqq M_1\|x^{(k)}-\bar{x}\|$$

for $x^{(k)}\in U(\bar{x})$ *and a constant* M_1 *with* $\|J(\bar{x})\|<M_1<1$, *and hence have*

(3.6) $$x^{(k+1)}\in U(\bar{x}).$$

Lemma 2 ([2, Lemma 4]). *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *an* $n\times n$ *matrix* $\Delta^2X^{(k)}$ *is invertible, and there exists a constant* L_4 *such that the inequality*

(3.7) $$\|(\Delta^2X^{(k)})^{-1}\|\leqq L_4\|d^{(k)}\|^{-1}$$

holds for sufficiently large k.

As easily seen, we obtain

(3.8) $$\Delta x^{(k+1)}=(J(\bar{x})-I)[\Delta x^{(k)}+d^{(k)}+(J(\bar{x})-I)^{-1}\xi(x^{(k+1)})],$$

from (2.4), by (2.2), (3.1) and (A.3). By (2.5), we have $D_{k+1}=\Delta X^{(k)}+D_k$, and, by (3.1),
$$d^{(k+i)}-d^{(k+i-1)}=(J(\bar{x})-I)d^{(k+i-1)}+\xi(x^{(k+i-1)}),$$

so that

$$\Delta X^{(k+1)}=J(\bar{x})\Delta X^{(k)}+Y(x^{(k)},\cdots,x^{(k+n)}) \tag{3.9}$$

follows from (2.5). We observe that, from Lemma 2, by (3.6), $\Delta^2 X^{(k+1)}$ is invertible for $x^{(k)}\in U(\bar{x})$. Hence, by writing

$$(\Delta^2 X^{(k+1)})^{-1}=\{(\Delta^2 X^{(k)})^{-1}-[I-(\Delta^2 X^{(k+1)})^{-1}(J(\bar{x})-I)\Delta^2 X^{(k)}](\Delta^2 X^{(k)})^{-1}\}(J(\bar{x})-I)^{-1},$$

and using (2.6) and (3.9), we see that

$$\begin{aligned}(\Delta^2 X^{(k+1)})^{-1}=\{&(\Delta^2 X^{(k)})^{-1}-(\Delta^2 X^{(k+1)})^{-1}[(J(\bar{x})-I)\Delta X^{(k)}\\&+Y(x^{(k+1)},\cdots,x^{(k+n+1)})](\Delta^2 X^{(k)})^{-1}\}(J(\bar{x})-I)^{-1}.\end{aligned} \tag{3.10}$$

4. Proof of Theorem 1. We shall prove Theorem 1. As may be seen by Remark 1 in § 2, we have $y^{(k)}\to\bar{x}$ as $k\to\infty$. Now, (2.3) gives

$$y^{(k+1)}-\bar{x}=d^{(k+1)}-\Delta X^{(k+1)}(\Delta^2 X^{(k+1)})^{-1}\Delta x^{(k+1)}. \tag{4.1}$$

Substituting (3.1), (3.8), (3.9) and (3.10) into (4.1), it yields

$$\begin{aligned}y^{(k+1)}-\bar{x}=&J(\bar{x})(y^{(k)}-\bar{x})+\xi(x^{(k)})\\&+p_1(x^{(k)},\cdots,x^{(k+n+1)})+p_2(x^{(k)},\cdots,x^{(k+n+2)})\\&+p_3(x^{(k)},\cdots,x^{(k+n+1)})+p_4(x^{(k)},\cdots,x^{(k+n+2)}),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}&p_1(x^{(k)},\cdots,x^{(k+n+1)})\\&\quad=-J(\bar{x})\Delta X^{(k)}(\Delta^2 X^{(k)})^{-1}[d^{(k)}+(J(\bar{x})-I)^{-1}\xi(x^{(k+1)})],\end{aligned} \tag{4.3}$$

$$\begin{aligned}&p_2(x^{(k)},\cdots,x^{(k+n+2)})\\&\quad=J(\bar{x})\Delta X^{(k)}(\Delta^2 X^{(k+1)})^{-1}[(J(\bar{x})-I)\Delta X^{(k)}\\&\qquad+Y(x^{(k+1)},\cdots,x^{(k+n+1)})](\Delta^2 X^{(k)})^{-1}(J(\bar{x})-I)^{-1}\Delta x^{(k+1)},\end{aligned} \tag{4.4}$$

$$\begin{aligned}&p_3(x^{(k)},\cdots,x^{(k+n+1)})\\&\quad=-Y(x^{(k)},\cdots,x^{(k+n)})(\Delta^2 X^{(k)})^{-1}(J(\bar{x})-I)^{-1}\Delta x^{(k+1)},\end{aligned} \tag{4.5}$$

$$\begin{aligned}&p_4(x^{(k)},\cdots,x^{(k+n+2)})\\&\quad=Y(x^{(k)},\cdots,x^{(k+n)})(\Delta^2 X^{(k+1)})^{-1}[(J(\bar{x})-I)\Delta X^{(k)}\\&\qquad+Y(x^{(k+1)},\cdots,x^{(k+n+1)})](\Delta^2 X^{(k)})^{-1}(J(\bar{x})-I)^{-1}\Delta x^{(k+1)}.\end{aligned} \tag{4.6}$$

Recall that $x^{(k+1)}\in U(\bar{x})$, provided $x^{(k)}\in U(\bar{x})$. Then, (3.2), (3.3) and (3.7) lead to

$$\|\xi(x^{(k+1)})\|\leqq L_1\|d^{(k+1)}\|^2\leqq L_1\|d^{(k)}\|^2, \tag{4.7}$$

$$\|Y(x^{(k+1)},\cdots,x^{(k+n+1)})\|\leqq L_2\|d^{(k+1)}\|^2\leqq L_2\|d^{(k)}\|^2 \tag{4.8}$$

and

$$\|(\Delta^2 X^{(k+1)})^{-1}\|\leqq L_4\|d^{(k+1)}\|^{-1}, \tag{4.9}$$

respectively. Since $d^{(k+1)}=\Delta x^{(k)}+d^{(k)}$, it follows from (3.8) that

$$\|\Delta x^{(k+1)}\|\leqq L_5\|d^{(k+1)}\|\leqq L_5\|d^{(k)}\| \tag{4.10}$$

for a constant L_5 chosen suitably. In (4.7), (4.8) and (4.10), we have used the fact (3.5) in Lemma 1.

Now, as for equalities (4.3)–(4.6), there exist constants L_6, L_7, L_8 and L_9 such that the following estimates (4.11)–(4.14) hold:

$$\|p_1(x^{(k)},\cdots,x^{(k+n+1)})\|\leqq L_6\|d^{(k)}\| \tag{4.11}$$

from (4.3), by (3.4), (3.7) and (4.7);

$$\|p_2(x^{(k)},\cdots,x^{(k+n+2)})\|\leqq L_7\|d^{(k)}\| \tag{4.12}$$

from (4.4), by (3.4), (3.7), (4.8), (4.9) and (4.10);

$$\|p_3(x^{(k)},\cdots,x^{(k+n+1)})\|\leqq L_8\|d^{(k)}\|^2 \tag{4.13}$$

from (4.5), by (3.3), (3.7) and (4.10);

(4.14) $$\|p_4(x^{(k)}, \cdots, x^{(k+n+2)})\| \leqq L_9 \|d^{(k)}\|^2$$

from (4.6), by (3.3), (3.4), (3.7), (4.8), (4.9) and (4.10).

Consequently, (4.2), together with (3.2) and (4.11)–(4.14), shows that

$$\|y^{(k+1)} - \bar{x}\| \leqq M \|y^{(k)} - \bar{x}\| + \varepsilon_k$$

holds with a constant M satisfying $\|J(\bar{x})\| < M < 1$, where

$$\varepsilon_k = (L_6 + L_7 + (L_1 + L_8 + L_9)\|d^{(k)}\|)\|d^{(k)}\| \to 0 \qquad \text{as } k \to \infty.$$

Thus we have proved Theorem 1, as desired.

The author would like to express his hearty thanks to Prof. H. Mine of Kyoto University for many valuable suggestions.

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