48. *Regularity of the 90-velocity Model of the Boltzmann Equation*

By Yasushi SHIZUTA,*) Machi MAEJI,**) Akemi WATANABE,*) and Shuichi KAWASHIMA*)

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In this paper, we continue the study of discrete velocity models of the Boltzmann equation. Our aim is to show the existence of infinitely many regular models that are invariant under the symmetry group $G=A_5\times I$. (Here A_5 stands for the icosahedron group and I denotes the group of order 2 generated by the central inversion.) To this end we show the regularity of the 90-velocity model. (For the definition of the regular discrete models, see [2], [3].) Once this is established, we are able to construct a series of regular discrete models by induction.

1. First we consider a dodecahedron with edge of length 4 and assume that the center of symmetry is set at the origin of $\boldsymbol{R}^3$. We look at the 5 inscribed cubes of the dodecahedron, say, I, II, $\cdots$, V. We let coincide the 3 axes of the 4-fold rotational symmetry of the cube I with x-, y-, and z-axes. We define $v_1, \cdots, v_6$ to be the centers of 6 surfaces of the cube I. Similarly, $v_7, \cdots, v_{12}, \cdots, v_{25}, \cdots, v_{30}$ are defined with respect to the cubes II, $\cdots$, V, respectively. We set $M_1=\{v_1, \cdots, v_{30}\}$. Note that the moduli of $v_1, \cdots, v_{30}$ are all the same. Next we define $v_{31}, \cdots, v_{42}$ to be the mid-points of 12 edges of the cube I. Observe that $v_1, \cdots, v_6, v_{31}, \cdots, v_{42}$ form the 18-velocity model studied in [3]. We define also $v_{43}, \cdots, v_{54}, \cdots, v_{79}, \cdots, v_{90}$ to be the mid-points of the edges of the cubes II, $\cdots$, V, respectively. Then the moduli of $v_{31}, \cdots, v_{90}$ are equal to each other. We set $M_2=\{v_{31}, \cdots, v_{90}\}$. Finally we define M to be $M_1\cup M_2$. This is the 90-velocity model. It is clear by construction that the symmetry group G associated with M is $A_5\times I$.

2. We enumerate the collisions of this model. Let $p^{(1)}$ and $p^{(2)}$ be the respective numbers of collisions of M_1 and M_2. Then, it is shown that $p^{(1)}=390$ and that $p^{(2)}=1{,}410$. Let $p^{(1,2)}$ be the number of mixed collisions of M_1 and M_2. The computation shows that $p^{(1,2)}=2{,}280$. Therefore, p, the total number of collisions of M is equal to 4,080. The collisions are classified by using the transformation group $G=A_5\times I$. Let $q^{(1)}$ and $q^{(2)}$ be the numbers of types of collisions of M_1 and M_2, respectively. Then, we have $q^{(1)}=7$ and $q^{(2)}=19$. We define similarly $q^{(1,2)}$ with respect to the mixed collisions of M_1 and M_2. It is shown that $q^{(1,2)}=18$. Hence, q, the total number of types of collisions of M is equal to 44. The space of summational

*) Department of Mathematics, Nara Women's University.
**) Central Research Laboratory, Mitsubishi Electric Corporation.

invariants of the 90-velocity model has dimension 8.

Remark. The 90-velocity model resembles the 32-velocity models of Cabannes studied in [4]. But, in the former model, the ratio of two different moduli of velocities equals $\sqrt{2}$. In other words, the energy of the particles with velocities belonging to M_2 is twice the energy of the particles with velocities belonging to M_1. This ratio is a more complicated number in the latter models. For this reason we do not use the 32-velocity model as the starting member in constructing another models with larger sizes.

3. To show the regularity of the 90-velocity model, it is enough to verify the stability condition because the irreducibility is easily seen. Actually we check the stability condition in the reduced form. (See [4].) For this purpose, we enumerate the P-sets and then we classify them by using the transformation group $G=A_5\times I$. (See [1] for details.) Let r be the total number of the P-sets of M in the strict sense. It is shown by using the computer that $r=40,829$. Let $1\leq i\leq 90$ and let $r^{(i)}$ be the number of P-sets in the strict sense, whose cardinalities are equal to i. Then we obtain, $r^{(3)}=33,320$, $r^{(4)}=4,920$, $r^{(5)}=1,074$, $r^{(6)}=1,070$, $r^{(7)}=240$, $r^{(8)}=105$, $r^{(9)}=60$, $r^{(10)}=6$, $r^{(15)}=24$, $r^{(18)}=10$. For other i's, we have $r^{(i)}=0$. Now let s be the number of equivalence classes of the P-sets in the strict sense. Then we obtain $s=408$. We denote by $s^{(i)}$ the number of classes whose cardinalities are equal to i. Here, $1\leq i\leq 90$. Then, $s^{(3)}=288$, $s^{(4)}=66$, $s^{(5)}=19$, $s^{(6)}=20$, $s^{(7)}=4$, $s^{(8)}=4$, $s^{(9)}=3$, $s^{(10)}=1$, $s^{(15)}=2$, $s^{(18)}=1$. We list up a system of the representatives of 408 classes and check the stability condition in the reduced form. All these computations are carried out by using the computer and we obtain finally the affirmative result. (We need also a basis of the space of summational invariants in computation. We use the 8 summational invariants issuing from the conservation of mass, momentum, and energy, as in the case of the 32-velocity models.)

Theorem 1. *The 90-velocity model is a regular discrete model.*

As a consequence of this theorem, we obtain the following theorem which was announced in [2].

Theorem 2. *Let $k\geq 2$ be an arbitrary integer and let $G=A_5\times I$. Then $\mathfrak{R}(G)$, the set of regular discrete models with the transformation group G, contains a model with k moduli of velocities.*

To prove Theorem 2, we employ almost the same arguments as in [3]. Finally we give the list of types of the mixed collisions. Other types and the coordinates for $v_1, \cdots, v_{90}$ are given in a companion paper [5]. For the notations such as E, M, I, see [4].

Classification of mixed collisions

type	number of collision	representative	E	M	I
C_{27}	240	$(v_1, v_{32}) \to (v_2, v_{31})$	$12(1+\tau)$	$12(1+\tau)$	$1/3$
C_{28}	120	$(v_1, v_{32}) \to (v_{27}, v_{88})$			
C_{29}	120	$(v_1, v_{32}) \to (v_{26}, v_{81})$	$12(1+\tau)$	$12(1+\tau)$	$(-1+2\tau)/3$
C_{30}	120	$(v_1, v_{32}) \to (v_{28}, v_{80})$			
C_{31}	240	$(v_1, v_{33}) \to (v_2, v_{34})$	$12(1+\tau)$	$4(1+\tau)$	$1/5$
C_{32}	60	$(v_1, v_{33}) \to (v_3, v_{31})$	$12(1+\tau)$	$4(1+\tau)$	$3/5$
C_{33}	60	$(v_1, v_{40}) \to (v_4, v_{39})$			
C_{34}	120	$(v_1, v_{47}) \to (v_{27}, v_{62})$	$12(1+\tau)$	$4(2+\tau)$	$(3+2\tau)/11$
C_{35}	120	$(v_1, v_{48}) \to (v_{24}, v_{59})$	$12(1+\tau)$	$4(3+4\tau)$	$(5-2\tau)/9$
C_{36}	120	$(v_1, v_{49}) \to (v_{11}, v_{38})$	$12(1+\tau)$	$4(2+3\tau)$	$(5+6\tau)/19$
C_{37}	120	$(v_1, v_{49}) \to (v_{13}, v_{87})$			
C_{38}	120	$(v_1, v_{49}) \to (v_{30}, v_{60})$	$12(1+\tau)$	$4(2+3\tau)$	$3(-3+4\tau)/19$
C_{39}	120	$(v_1, v_{50}) \to (v_{10}, v_{38})$	$12(1+\tau)$	4	$(11-6\tau)/19$
C_{40}	120	$(v_1, v_{50}) \to (v_{21}, v_{61})$			
C_{41}	120	$(v_1, v_{50}) \to (v_{17}, v_{78})$	$12(1+\tau)$	4	$2(-1+6\tau)/19$
C_{42}	120	$(v_1, v_{50}) \to (v_9, v_{38})$	$12(1+\tau)$	$8(1+\tau)$	$1/2$
C_{43}	120	$(v_1, v_{52}) \to (v_{19}, v_{81})$	$12(1+\tau)$	8	0
C_{44}	120	$(v_1, v_{54}) \to (v_{26}, v_{74})$	$12(1+\tau)$	$8(2+3\tau)$	0

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