# 33. Wave Forms on O(1, q+1) and associated Dirichlet Series

By Koichi TAKASE
Department of Mathematics, Tokyo Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., March 12, 1986)

**§1. Wave forms.** We study wave forms on $O(1, q+1)$ and their Dirichlet series of two types. Details are described in [5].

Let $S_0 \in M(q, \boldsymbol{Q})$ be a symmetric positive definite matrix of size $q>0$, and put $S=\begin{pmatrix} & & 1 \\ & S_0 & \\ 1 & & \end{pmatrix}$ which is a symmetric matrix of signature $(1, q+1)$. Let $\tilde{G}$ be the reductive algebraic group over $\boldsymbol{Q}$ whose $\boldsymbol{Q}$-rational points are $\tilde{G}_{\boldsymbol{Q}}=\{g \in \mathrm{GL}(q+2, \boldsymbol{Q}) \mid {}^t gSg=\nu(g)S \text{ for } \nu(g) \in \boldsymbol{Q}^\times\}$. Each element $g \in \tilde{G}$ is denoted by $g=\begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}\begin{matrix} \}1 \\ \}q \\ \}1 \end{matrix}$. The semi-simple part of $\tilde{G}$ is $G=\{g \in \tilde{G} \mid \nu(g)=1\}$. Put $P=\left\{\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & j \end{pmatrix} \in G\right\}$ which is a minimal parabolic subgroup of $G$ defined over $\boldsymbol{Q}$, and $P$ has the decomposition $P=NAM$ where

$$N=\left\{\begin{pmatrix} 1 & b & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \in G\right\}, \quad A=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & j \end{pmatrix} \in G\right\}, \quad M=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G\right\}.$$

We have an isomorphism $n: \boldsymbol{Q}^q \xrightarrow{\sim} N_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$ which is given by

$$n(x)=\begin{pmatrix} 1 & -{}^t xS_0 x & -(1/2){}^t xS_0 x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\tilde{G}_0$ be the algebraic group over $\boldsymbol{Q}$ whose $\boldsymbol{Q}$-rational points are $\tilde{G}_{0,\boldsymbol{Q}}=\{e \in \mathrm{GL}(q, \boldsymbol{Q}) \mid {}^t eS_0 e=\nu(e)S_0 \text{ for } \nu(e) \in \boldsymbol{Q}^\times, \det(e)=\nu(e)^{q/2} \text{ if } q \text{ is even}\}$. The adelization of $\tilde{G}$ (resp. $G$, $P$, etc.) over $\boldsymbol{Q}$ is denoted by $\tilde{G}_A$ (resp. $G_A$, $P_A$, etc.).

Let $L_0$ be a $\boldsymbol{Z}$-lattice in $\boldsymbol{Q}^q$ which is maximal integral with respect to $S_0$, and put $L=\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{Q}^{q+2} \mid x \in \boldsymbol{Z}, y \in L_0, z \in \boldsymbol{Z}\right\}$. Then $L$ is a $\boldsymbol{Z}$-lattice in $\boldsymbol{Q}^{q+2}$ which is maximal integral with respect to $S$. Put $\tilde{K}_p=\{g \in \tilde{G}_p \mid g(L_p)=L_p\}$ for each primes $p$ of $\boldsymbol{Q}$ with $L_p=L\otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$, and put

$$\tilde{K}_\infty=\left\{g \in \tilde{G}_\infty \mid {}^t g\begin{pmatrix} 1 & & \\ & S_0 & \\ & & 1 \end{pmatrix}g=\begin{pmatrix} 1 & & \\ & S_0 & \\ & & 1 \end{pmatrix}\right\}.$$

Then $\tilde{K}=\prod_{p\leq\infty} \tilde{K}_p$ is a compact subgroup of $\tilde{G}_A$. Put $\tilde{U}_p=\{e \in \tilde{G}_{0,p} \mid e(L_{0,p})=L_{0,p}\}$

for each primes $p$ of $\boldsymbol{Q}$ with $L_{0,p}=L_0\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$, and $\tilde{U}_f=\prod_{p<\infty}\tilde{U}_p$ is an open compact subgroup of the finite part $\tilde{G}_{0,f}$ of $\tilde{G}_{0,A}$. We put $K=\tilde{K}\cap G_A$.

Take a continuous unitary character $\omega$ of the idèle class group $\boldsymbol{Q}_A^\times/\boldsymbol{Q}^\times$, and a complex number $\rho$. A wave form of type $(\omega, \rho)$ is a continuous $\boldsymbol{C}$-valued function $\Phi$ on $\tilde{G}_A$ satisfying the following four conditions; 1) $\Phi(x\gamma gk)=\omega(x)\Phi(g)$ for all $x\in Q_A^\times$, $\gamma\in\tilde{G}_{\boldsymbol{Q}}$, $k\in\tilde{K}$, 2) $\Phi$ is real analytic with respect to the infinite part of $\tilde{G}_A$, 3) $D\Phi=\{\rho^2-(q/2)^2\}\Phi$ where $D=2(q-1)\times$ the Casimir element for $\mathrm{Lie}(G_\infty)\otimes_{\boldsymbol{R}}\boldsymbol{C}$, 4) slowly increasing. We denote by $A(\omega, \rho)$ the $\boldsymbol{C}$-vector space of the wave forms of type $(\omega, \rho)$, which is finite dimensional. Each $\Phi\in A(\omega, \rho)$ has the Fourier expansion

$$\Phi(n(x)g)=\sum_{u\in\boldsymbol{Q}^q}\Phi_u(g)\Lambda({}^tuS_0x)$$

where $\Lambda$ is the continuous unitary character of $\boldsymbol{Q}_A/\boldsymbol{Q}$ such that $\Lambda_\infty(x)=\exp(-2\pi\sqrt{-1}x)$. For each $0\neq u\in\boldsymbol{Q}^q$, we have $\Phi_u(g)=C_u(\Phi, g_f)W_{\rho,u}(g_\infty)$ where $g_f$ (resp. $g_\infty$) is the finite part (resp. infinite part) of $g\in\tilde{G}_A$ and $W_{\rho,u}$ is a real analytic function on $\tilde{G}_\infty$ such that

$$W_{\rho,u}\left(\begin{pmatrix} y & & \\ & 1 & \\ & & y^{-1}\end{pmatrix}\right)=K_\rho\left(4\pi\left(\frac{1}{2}{}^tuS_0u\right)^{1/2}y\right)\left\{4\pi\left(\frac{1}{2}{}^tuS_0u\right)^{1/2}y\right\}^{q/2}\qquad (0<y\in\boldsymbol{R})$$

with the modified Bessel function $K_\rho(x)$ (see [3] p. 66). By virtue of the Iwasawa decomposition of $\tilde{G}_\infty$, the function $W_{\rho,u}$ is uniquely determined. We denote by $S(\omega, \rho)=\{\Phi\in A(\omega, \rho)\,|\,\Phi_0(g)=0$ for all $g\in\tilde{G}_A\}$ the $\boldsymbol{C}$-vector space of the cuspidal wave forms. We notice that $A(\omega, \rho)\neq 0$ only if $\omega=|\ |_A^\sigma$ with the absolute value $|\ |_A$ of the idèles and a purely imaginary complex number $\sigma$.

§ 2. **Mellin transformation.** For a wave form $\Phi\in A(\omega, \rho)$ and a $\boldsymbol{C}$-valued continuous function $\phi$ on $\tilde{G}_{0,\boldsymbol{Q}}\backslash\tilde{G}_{0,f}/\tilde{U}_f$, we put

$$Z(s\,;\,\Phi, \phi)=2^{(1/2)q-1}\cdot\Gamma_{q,\rho}(s)$$
$$\times\sum_{0\neq u\in\boldsymbol{Q}^q}\sum_e C_u\left(\Phi, \begin{pmatrix}\nu(e) & & \\ & e & \\ & & 1\end{pmatrix}\right)\phi(e)|\nu(e)|_f^{(1/2)(s-\sigma)}\left(\frac{1}{2}{}^tuS_0u\right)^{-(1/2)s}$$

where $\Gamma_{q,\rho}(s)=(2\pi)^{-s}\Gamma((1/2)(s+q/2+\rho))\Gamma((1/2)(s+q/2-\rho))$ is a product of $\Gamma$-functions, $\sum_e$ is the summation over the representatives of $\tilde{G}_{0,\boldsymbol{Q}}\backslash\tilde{G}_{0,f}/\tilde{U}_f$ which is a finite set, and $|\ |_f$ is the finite part of $|\ |_A$. Then $Z(s\,;\,\Phi, \phi)$ is a Dirichlet series which converges absolutely for $\mathrm{Re}(s)\gg 0$. By means of the Mellin transformation of $\Phi$, we have

**Theorem 1.** *$Z(s\,;\,\Phi, \phi)$ has a meromorphic continuation to the whole s-plane with a functional equation $Z(s\,;\,\Phi, \phi)=Z(-s\,;\,\check{\Phi}, \check{\phi})$ where $\check{\Phi}(g)=\Phi(g)\omega(\nu(g)^{-1})$ and $\check{\phi}(e)=\phi(\nu(e)^{-1}e)$. Moreover $Z(s\,;\,\Phi, \phi)$ is holomorphic except for the possible poles at $s=\pm q/2\pm\rho$ of order at most 1 if $\rho\neq 0$ (2 if $\rho=0$). $Z(s\,;\,\Phi, \phi)$ is entire if $\Phi$ is cuspidal.*

**Remark 1.** When $\phi$ is the characteristic function of $\tilde{G}_{0,\boldsymbol{Q}}\tilde{U}_f$ in $\tilde{G}_{0,f}$, Theorem 1 gives the meromorphic continuation and the functional equation of the Dirichlet series

$$\sum_{0\neq u\in \boldsymbol{Q}^q} C_u(\Phi, 1)\left(\frac{1}{2}{}^t u S_0 u\right)^{-(1/2)s}$$

which is treated by Maass [2].

§3. **Rankin-Selberg method.** Throughout this section, we suppose that $S_0=\begin{pmatrix} S_0' & 0 \\ 0 & S_0'' \end{pmatrix}\begin{matrix}\}m \\ \}q-m\end{matrix}$ $(0<m<q)$ and that the $\boldsymbol{Z}$-lattice $L_0$ has an orthogonal splitting $L_0=L_0'\oplus L_0''$ $(L_0'\subset \boldsymbol{Q}^m,\ L_0''\subset \boldsymbol{Q}^{q-m})$. Put $S'=\begin{pmatrix} & & 1 \\ & S_0' & \\ 1 & & \end{pmatrix}$ and define the algebraic groups $G'$, $P'$, $N'$, $A'$, and $M'$ with respect to $S'$ as in §1. The algebraic group $G'$ is identified with an algebraic subgroup of $G$ via the mapping

$$\begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}\begin{matrix}\}1 \\ \}m \\ \}1\end{matrix} \longmapsto \begin{pmatrix} a & b & 0 & c \\ d & e & 0 & f \\ 0 & 0 & 1 & 0 \\ h & i & 0 & j \end{pmatrix}\begin{matrix}\}1 \\ \}m \\ \}q-m \\ \}1\end{matrix}.$$

The compact subgroup $K'$ of $G_A'$ is defined as in §1 by the $\boldsymbol{Z}$-lattice $L_0'$. Take and fix a $\boldsymbol{C}$-valued continuos function $\phi$ on $M_{\boldsymbol{Q}}'A_A'N_A'\backslash G_A'/K'$.

We put $\theta(g, s)=|a|_A^s$ for $s\in \boldsymbol{C}$ and $g=\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & j \end{pmatrix}k\in G_A'=P_A'K'$ with $k\in K'$. Then the Eisenstein series associated with $\phi$ and the pair $(G', P')$ is defined by

$$E(\phi;\ s, g)=\sum_{\gamma\in P_{\boldsymbol{Q}}'\backslash G_{\boldsymbol{Q}}'} \phi(\gamma g)\theta(\gamma g, s+m/2)$$

which converges absolutely for $\mathrm{Re}(s)>m/2$ and all $g\in G_A'$. It has a meromorphic continuation to the whole $s$-plane with a functional equation (see Arthur [1]).

For a wave form $\Phi\in A(\omega, \rho)$, we put for each $0\neq u\in \boldsymbol{Q}^{q-m}$

$$C_{u,\phi}(\Phi)=\sum_h C_{\binom{0}{u}}(\Phi, h)\phi(h)$$

where $\sum_h$ is the summation over the representatives of $M_{\boldsymbol{Q}}'A_A'N_A'\backslash G_A'/K'$ which is a finite set. Then we have

**Theorem 2.** *For any cuspidal wave form $\Phi\in S(\omega, \rho)$ we have the following Rankin-Selberg type identity*

$$\int_{G_{\boldsymbol{Q}}'\backslash G_A'} \Phi(g)E(\phi;\ s-m/2, g)dg$$

$$=2^{(1/2)q-1}\cdot\Gamma_{q,\rho}(s)\times\sum_{0\neq u\in \boldsymbol{Q}^{q-m}} C_{u,\phi}(\Phi)\left(\frac{1}{2}{}^t u S_0'' u\right)^{-(1/2)s} \qquad (\mathrm{Re}(s)\gg 0).$$

When $\phi|_{M_f'}$ is the characteristic function of $M_{\boldsymbol{Q}}'(M_f'\cap K')$ in $M_f'$, Theorem 2 gives

**Corollary 1.** *For any cuspidal wave form $\Phi\in S(\omega, \rho)$, the Dirichlet series*

$$\sum_{0\neq u\in \boldsymbol{Q}^{q-m}} C_{\binom{0}{u}}(\Phi, 1)\left(\frac{1}{2}{}^t u S_0'' u\right)^{-(1/2)s} \qquad (\mathrm{Re}(s)\gg 0)$$

*is meromorphic on $\boldsymbol{C}$.*

If $m=1$, the Eisenstein series $E(1; s, g)$ has a simple functional equation, and we have

**Corollary 2.** *Suppose $m=1$. Then for each cuspidal wave form $\Phi \in S(\omega, \rho)$,*

$$\tilde{Z}(s, \Phi)=Z(2s)\cdot\Gamma_{q,\rho}(s)\times\sum_{0\neq u\in\boldsymbol{Q}^{q-1}} C_{\binom{0}{u}}(\Phi, 1)\left(\frac{1}{2}{}^tuS_0''u\right)^{-(1/2)s} \qquad (\mathrm{Re}\,(s)\gg 0)$$

*is meromorphic on* $\boldsymbol{C}$ *with a functional equation*

$$\tilde{Z}(1-s, \Phi)=\mathrm{vol}\,(\boldsymbol{R}/L_0')\sqrt{S_0'/2}\,\tilde{Z}(s, \Phi),$$

*where* $Z(s)=\pi^{-(1/2)s}\Gamma(s/2)\zeta(s)$.

**Remark 2.** If $m=q-1$ and a wave form $\Phi\in S(\omega, \rho)$ is Hecke eigen, then the Dirichlet series

$$\sum_{0\neq u\in\boldsymbol{Q}} C_{u,\phi}(\Phi)\left(\frac{1}{2}{}^tuS_0''u\right)^{-(1/2)s}$$

corresponds to the standard $L$-function associated with $\Phi$ in the sense of Langlands (see Sugano [4, § 3]).

**Remark 3.** The wave forms on O(1, 2) (resp. O(1, 3)) correspond to automorphic forms on GL (2) over $\boldsymbol{Q}$ (resp. an imaginary quadratic field $F$) via the isogeny mapping $\mathrm{SO}\,(1, 2)_{\boldsymbol{R}}\sim\mathrm{SL}\,(2, \boldsymbol{R})$ (resp. $\mathrm{SO}\,(1, 3)_{\boldsymbol{R}}\sim\mathrm{SL}\,(2, \boldsymbol{C})$). Then the Dirichlet series defined in § 1 corresponds to the standard $L$-function $\sum_{0<n\in\boldsymbol{Z}} C(f, n)n^{-s}$ (resp. $\sum_{\mathfrak{a}\subset\mathfrak{O}_F} C(f, \mathfrak{a})N(\mathfrak{a})^{-s}$) associated with the automorphic form $f$ on GL (2) over $\boldsymbol{Q}$ (resp. $F$). In the case of O(1, 3), the Dirichlet series defined in § 2 corresponds to the Dirichlet series

$$\sum_{0<n\in\boldsymbol{Z}} C(f, n\mathrm{O}_F)n^{-s}.$$

## References

[1] J. Arthur: Eisenstein series and the trace formula. Proc. Sympos. Pure Math., vol. 33, Part I, Amer. Math. Soc. Providence R. I., 253–274 (1979).

[2] H. Maass: Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen. Abh. Math. Sem. Univ. Hamburg, **16**, 72–100 (1949).

[3] W. Magnus *et al.*: Formulas and Theorems for the Special Functions of Mathematical Physics. Springer-Verlag (1966).

[4] T. Sugano: On Dirichlet series attached to holomorphic cusp forms on SO(2, $q$) (to appear).

[5] K. Takase: On the wave forms on O(1, $q+1$) and the associated Dirichlet series (preprint, 1985).