31. On the Crepant Blowing-Ups of Canonical Singularities and Its Application to Degenerations of Surfaces

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Let X be a normal algebraic variety over $\boldsymbol{C}$, and let D be a Weil divisor on it. We would like to know when the sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{R}(D) := \oplus_{m \geqq 0} \mathcal{O}_X(mD)$ is finitely generated, where the $\mathcal{O}_X(mD)$ are reflexive sheaves of rank 1 corresponding to the mD. It is equivalent to saying that there exists a projective morphism $f: X' \to X$ which is an isomorphism in codimension 1 and such that the strict transform D' of D on X' is $\boldsymbol{Q}$-Cartier and f-ample. The problem is trivial in case $\dim X = 2$; f must be an isomorphism and the condition for the finite generatedness is simply that D is $\boldsymbol{Q}$-Cartier. It is well known that a normal surface singularity X is (analytically) $\boldsymbol{Q}$-*factorial*, i.e., an arbitrary (analytic) Weil divisor on X is $\boldsymbol{Q}$-Cartier, if and only if X is a rational singularity. In this paper we announce a partial generalization of this fact to 3-dimensional case. (We refer the reader to [3] for definitions concerning minimal models.)

Theorem 1. *Let X be a 3-dimensional normal algebraic variety over $\boldsymbol{C}$ which has at most canonical singularities, and let D be a Weil divisor on it. Then $\mathcal{R}(D)$ is finitely generated.*

We note that a rational Gorenstein singularity is canonical. The theorem is proved in the following way. Let X be as in Theorem 1 and let $\mu: Y \to X$ be a desingularization. Then we can write $K_Y = \mu^* K_X + \sum_j a_j F_j$ with $a_j \geqq 0$ by definition, where the F_j are exceptional divisors of μ. We define $e(X)$ as the number of divisors F_j for which μ is *crepant*, i.e., $a_j = 0$ (it is easy to see that $e(X)$ does not depend on the choice of μ). For example, $e(X) = 0$ if and only if X has at most terminal singularities. We define also $\sigma(X) := \dim_{\boldsymbol{Q}} Z_2(X)_{\boldsymbol{Q}} / \mathrm{Div}(X)_{\boldsymbol{Q}}$, where $Z_2(X)_{\boldsymbol{Q}}$ and $\mathrm{Div}(X)_{\boldsymbol{Q}}$ are groups of $\boldsymbol{Q}$-divisors and $\boldsymbol{Q}$-Cartier divisors, respectively (one can prove that $\sigma(X)$ is finite). Thus X is $\boldsymbol{Q}$-factorial if and only if $\sigma(X) = 0$. Our theorem is proved by induction on $e(X)$ and $\sigma(X)$ in the category consisting of varieties X' with projective birational morphisms $f: X' \to X$ which are *crepant*, i.e., $K_{X'} = f^* K_X$; e.g., an isomorphism in codimension 1 is crepant, since K_X is $\boldsymbol{Q}$-Cartier. Theorem 1 in case $e(X) = 0$ is proved by using Brieskorn's flips as in [5]. The termination of log-flips in case $e(X') = n$ produces the existence of the log-flip in case $e(X') = \mathrm{n} + 1$ (cf. [3]). In the course of the proof, the concept of the *sectional decomposition*, which is a rather trivial generalization of the Zariski decomposition for surfaces (cf.

[7]), plays an essential role. We employ a technique developed in [2] to deal with the difficulty concerning $\boldsymbol{R}$-divisors which inevitably appear in higher dimensional sectional decompositions (cf. [1]). More precisely, we prove following Lemmas 2 to 5 in our inductive argument.

Lemma 2. *Let X be a 3-dimensional variety with $\boldsymbol{Q}$-factorial canonical singularities such that $e(X)\geqq 1$. Then there exists a projective birational morphism $f: X_1\to X$ such that* (i) *X_1 has at most $\boldsymbol{Q}$-factorial canonical singularities,* (ii) *the exceptional locus of f is a prime divisor, and* (iii) *f is crepant.*

Lemma 3. *There is a function $b: \boldsymbol{N}\times(\boldsymbol{N}\cup\{0\})\to\boldsymbol{N}$ such that*

$$b(r, e)Z_2(X)\subset \mathrm{Div}\,(X)$$

for an arbitrary 3-dimensional variety X with at most $\boldsymbol{Q}$-factorial canonical singularities of index r and $e=e(X)$.

Lemma 4. *Let $\varphi: X\to Z$ be a projective morphism of 3-dimensional varieties and let D be a Cartier divisor on X. Assume that* (a) *X has at most $\boldsymbol{Q}$-factorial canonical singularities,* (b) *φ is an isomorphism in codimension 1,* (c) $\dim N^1(X/Z)=1$, *and* (d) *$(K_X\cdot C)=0$ and $(D\cdot C)<0$ for all curves C on X such that $\varphi(C)$ is a point. Then there exists a projective morphism $\varphi^+: X^+\to Z$ which satisfies the following conditions.* (i) *X^+ has at most $\boldsymbol{Q}$-factorial canonical singularities,* (ii) *φ^+ is an isomorphism in codimension 1,* (iii) $\dim N^1(X^+/Z)=1$, *and* (iv) *D^+ being the strict transform of D, $(K_{X^+}\cdot C^+)=0$ and $(D^+\cdot C^+)>0$ for all curves C^+ such that $\varphi^+(C^+)$ is a point.*

We call the prccedure to obtain φ^+ from φ the *log-flip* with respect to D. Let $f: X\to S$ be a projective surjective morphism with connected fibers such that $\dim X=3$, X has at most $\boldsymbol{Q}$-factorial canonical singularities, and that $\mathrm{cl}\,(K_X)=0$ in $N^1(X/S)$. A Weil divisor D on X is called *f-movable* if $f_*\mathcal{O}_X(D)\neq 0$ and if the cokernel of the natural homomorphism $f^*f_*\mathcal{O}_X(D)\to\mathcal{O}_X(D)$ has a support of codimension$\geqq 2$. We let $\overline{\mathrm{Apl}}\,(X/S)$, $\overline{\mathrm{Big}}\,(X/S)$ and $\overline{\mathrm{Mov}}\,(X/S)$ denote closed convex cones in $N^1(X/S)$ generated by the numerical classes of f-ample, f-big and f-movable divisors, and $\mathrm{Apl}\,(X/S)$, $\mathrm{Big}\,(X/S)$ and $\mathrm{Mov}\,(X/S)$ their interiors, respectively. The cone $\overline{\mathrm{Apl}}\,(X/S)\cap\mathrm{Big}\,(X/S)$ is locally polyhedral in $\mathrm{Big}\,(X/S)$ by Cone Theorem (cf. [3]). A log-flip leaves $\overline{\mathrm{Mov}}\,(X/S)\cap\mathrm{Big}\,(X/S)$ stable, while $\overline{\mathrm{Apl}}\,(X/S)\cap Big\,(X/S)$ is transformed to a neighboring cone $\overline{\mathrm{Apl}}\,(X^+/S)\cap\mathrm{Big}\,(X^+/S)$. The *sectional decomposition* of an f-big $\boldsymbol{R}$-divisor D relative to f, which always exists as far as X is $\boldsymbol{Q}$-factorial, is an expression $D=M+F$ in $Z_2(X)_{\boldsymbol{R}}$ such that $\mathrm{cl}\,(M)\in\overline{\mathrm{Mov}}\,(X/S)$, $F\geqq 0$, and that the natural homomorphisms $f_*\mathcal{O}_X([mM])\to f_*\mathcal{O}_X([mD])$ are bijective for all $m\in\boldsymbol{N}$.

Lemma 5. *Let $f: X\to S$ be a projective surjective morphism of varieties with connected fibers and let M be an $\boldsymbol{R}$-divisor on X. Assume the following conditions:* (a) *$\dim X=3$ and X has at most $\boldsymbol{Q}$-factorial canonical singularities of index r and $e=e(X)$,* (b) *$\mathrm{cl}\,(K_X)=0$ in $N^1(X/S)$,* (c) *M is f-big and $\mathrm{cl}\,(\mathrm{M})\in\overline{\mathrm{Mov}}\,(X/S)$, and* (d) *$D:=\ulcorner\mathrm{M}\urcorner\in 2b(r, e)\cdot Z_2(X)$. Then there* *does not exist an infinite sequence of log-flips with respect to the strict transforms of D.*

The following theorems are immediate applications of Theorem 1.

Theorem 6. *Let $f: X \to S$ be a projective surjective morphism with connected fibers and let D be a Weil divisor on X. Assume that* $\dim X=3$, *X has at most canonical singularities,* $\mathrm{cl}(K_X)=0$ *in* $N^1(X/S)$, *and that D is f-big. Then the sheaf of graded* $\mathcal{O}_S$*-algebras* $\mathcal{R}(X/S, D) := \oplus_{m \geqq 0} f_* \mathcal{O}_X(mD)$ *is finitely generated.*

Theorem 7. *Let X_1 and X_2 be two **Q**-factorial terminal good minimal models of dimension* 3 *which are birationally equivalent. Then they are joined by a sequence of log-flips.*

A singularity of a 3-dimensional normal variety Z is called *flipping* if it comes from a flipping contraction $\varphi: X \to Z$ from a variety X with terminal singularities (cf. [3]). The existence of minimal models for algebraic 3-folds follows if the existence of the flips is proved, and the latter is equivalent to the finite generatedness of $\mathcal{R}(K_Z)$ for flipping singularities of dimension 3. Theorem 1 gives a sufficient condition for this to hold; we construct a double covering $\pi: \tilde{Z} \to Z$ by using a section s of $\mathcal{O}_Z(-2K_Z)$. If $\tilde{Z}$ is a canonical singularity, then the finite generatedness of $\mathcal{R}(\pi^* K_Z)$ implies that of $\mathcal{R}(K_Z)$. In this way we obtain the following corollary to Theorem 1.

Corollary 8. *Let Z be a flipping singularity of dimension* 3 *and assume one of the following conditions:* (a) *there exists a Weil divisor S on Z with* $\mathcal{O}_Z(S) \simeq \mathcal{O}_Z(-K_Z)$ *which has at most rational singularities, or* (b) *the double covering of a hyperplane section of Z constructed by using the restriction of a section of* $\mathcal{O}_Z(-2K_Z)$ *has at most elliptic singularities. Then* $\mathcal{R}(K_Z)$ *is finitely generated.*

Finally, by using the criteria in Corollary 8, we obtain an alternative proof of the following (slightly generalized) theorem of Tsunoda [6] (Shokurov and Mori also announced to have their proofs in private letters).

Theorem 9. *Let $f: X \to S$ be a projective surjective morphism of smooth varieties with connected fibers such that* $\dim X=3$ *and* $\dim S=1$. *Assume that singular fibers of f are reduced and simple normal crossing while smooth fibers have non-negative Kodaira dimension. Then there exists a minimal model $f': X' \to S$ of f,* i.e., *f' is a projective surjective morphism which is birationally equivalent to f, X' has at most **Q**-factorial terminal singularities, and that $K_{X'}$ is f'-nef. In particular, smooth fibers of f' are minimal models of corresponding fibers of f.*

By applying Nakayama's theory [4], we can extend our results to the case where the base space is a complex analytic space; X may be a germ of an analytic space in Theorem 1 and S a disc in Theorem 9.

References

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