26. *On Some Properties of Set-dynamical Systems*

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1. Introduction. In [2], the author investigated self-similar sets, including classical singular curves like Peano's and Koch's, as the invariant sets under several contraction mappings.

In this paper, we shall treat such a peculiar set as the fixed point of a certain set-dynamical system.

Let X be a complete metric space with a metric d. The power set 2^X of all subsets of X forms a partially ordered set under set-inclusion in a natural way, that is, $x \leq y$ means x is a subset of y. Moreover, 2^X is a complete lattice with operations join "+" (set-union) and meet " · " (set-intersection). Let $C(X)$ be a subcollection of 2^X of all non-empty compact subsets of X, which is itself a partially ordered set under the same inclusion relation. Since $C(X) \not\ni \phi$ (empty set), $C(X)$ is not a lattice but a join-semilattice with the binary relation "+".

It is known that $C(X)$ is a complete metric space equipped with the Hausdorff metric:

$$d_H(x, y) = \max(\inf\{\varepsilon > 0,\ N_\varepsilon(x) \geq y\},\ \inf\{\varepsilon > 0,\ N_\varepsilon(y) \geq x\})$$

where $N_\varepsilon(x) \in 2^X$ is an ε-neighbourhood of the set x. Moreover, if X is compact, $C(X)$ becomes also a compact metric space [4]. Note that the mapping $i: X \to C(X)$, which maps p into $\{p\}$, is an isometry.

2. Induced mappings. A mapping $F: C(X) \to C(X)$ is said to be order-preserving provided that $x \leq y$ implies $F(x) \leq F(y)$; a join-endomorphism provided that $F(x+y) = F(x) + F(y)$ for all $x, y \in C(X)$. Let $\mathcal{F}$ consist of all continuous, order-preserving join-endomorphisms defined on $C(X)$; and let $F \leq G$ mean that $F(x) \leq G(x)$ for every $x \in C(X)$. Then $\mathcal{F}$ becomes a join-semilattice with operation "+", that is, $(F+G)(x)$ means $F(x) + G(x)$ for every $x \in C(X)$.

Now let $f: X \to X$ be a continuous mapping. Since the image of $x \in C(X)$ under f is plainly compact, we can define the *induced mapping* $f^*: C(X) \to C(X)$ in a natural way. Note that $(f \circ g)^* = f^* \circ g^*$ for any continuous self-mappings f, g. It is obvious that any induced mapping is contained in $\mathcal{F}$.

A self-mapping h defined on a metric space (E, δ) is said to satisfy the *condition* ψ provided that

$$\delta(h(x), h(y)) \leq \psi(\delta(x, y)) \qquad \text{for every } x, y \in E,$$

where $\psi(t)$ is a non-decreasing right-continuous real-valued function defined on $[0, \infty)$ satisfying $\psi(0) = 0$. Then we have:

Proposition 1. *Suppose that $f: X \to X$ satisfies the condition ψ. Then the induced mapping $f^*: C(X) \to C(X)$ also satisfies the same condition ψ.*

Proof. It is easily seen that $N_{\psi(\delta)}(f(x)) \geq f(N_\delta(x))$ for all $x \in C(X)$ and $\delta \geq 0$. Put $s = d_H(x, y)$ for brevity. Then for any $\varepsilon > 0$, we have $y \leq N_{s+\varepsilon}(x)$ and therefore $f(y) \leq f(N_{s+\varepsilon}(x)) \leq N_{\psi(s+\varepsilon)}(f(x))$. Similarly $f(x) \leq N_{\psi(s+\varepsilon)}(f(y))$. Thus, by definition, $d_H(f^*(x), f^*(y)) \leq \psi(s+\varepsilon)$. Taking $\varepsilon \to 0+$, we get the required inequality. □

Proposition 2. *Suppose that the induced mapping f_j^* satisfies the condition ψ_j for $1 \leq j \leq m$. Then the mapping $F = f_1^* + \cdots + f_m^* \in \mathcal{F}$ satisfies the condition $\psi(t) = \max_j \psi_j(t)$.*

The proof is straightforward.

A mapping satisfying the condition ψ, where $\psi(t) < t$ for any $t > 0$, is called a ψ-contraction. Suppose now that $f_1, \cdots, f_m$ are all ψ-contractions on X. Then, by Propositions 1 and 2, the mapping $F = f_1^* + \cdots + f_m^*$ is a ψ-contraction on $C(X)$. Therefore F has a unique fixed point K in $C(X)$, in other words, K is a unique non-empty compact subset of X satisfying the equality $K = f_1(K) + \cdots + f_m(K)$. This gives fairly simple another proof for the existence and uniqueness of the invariant set under several contractions discussed in [2].

3. Regular mappings. Given a mapping $F \in \mathcal{F}$, we will associate two mappings $L_F, R_F: C(X) \to 2^X$ as follows:

$$L_F(x) = \limsup_{n \to \infty} F^n(x) \quad \text{and} \quad R_F(x) = \text{closure of } \sum_{n \geq 0} F^n(x).$$

A mapping F is said to be *regular* provided that $R_F(x) \in C(X)$ for every $x \in C(X)$. Note that $R_F(x) \in C(X)$ implies $L_F(x) \in C(X)$. If the space X is compact, every $F \in \mathcal{F}$ must be regular.

Proposition 3. *Every ψ-contraction $F \in \mathcal{F}$ is regular. Moreover, R_F belongs to $\mathcal{F}$ and satisfies the condition $\psi(t) = t$.*

Proof. Put $x_n = F^n(x)$ for brevity. For any $\varepsilon > 0$, define a sufficiently large integer N such that $\psi^N(d_H(x_0, x_1)) \leq \varepsilon - \psi(\varepsilon)$. Since

$$d_H(x_N, x_{N+1}) \leq \psi^N(d_H(x_0, x_1)) \leq \varepsilon - \psi(\varepsilon),$$

we have inductively

$$d_H(x_N, x_{N+k}) \leq d_H(x_N, x_{N+1}) + d_H(x_{N+1}, x_{N+k}) \leq \varepsilon - \psi(\varepsilon) + \psi(d_H(x_N, x_{N+k-1})) \leq \varepsilon$$

for any $k \geq 1$. (This means that $\{x_n\}$ is a Cauchy sequence.) Let $\gamma(x)$, the measure of noncompactness of x [3], be $\inf\{\varepsilon > 0;\, x$ can be covered by a finite number of sets of diameter less than or equal to $\varepsilon\}$. Then,

$$\gamma\Big(\sum_{n \geq 0} x_n\Big) \leq \gamma\Big(\sum_{n \geq N} x_n\Big) \leq \gamma(N_\varepsilon(x_N)) \leq \gamma(x_N) + 2\varepsilon = 2\varepsilon.$$

Since ε is arbitrary, this implies that the set $\sum_{n \geq 0} x_n$ is pre-compact. Finally, it is easily seen that

$$d_H(R_F(x), R_F(y)) \leq \sup_n d_H(F^n(x), F^n(y)) = d_H(x, y). \qquad \square$$

4. Inhomogeneous equations. Under these preparations, we will give our main theorems.

Theorem 1. *Suppose that $F \in \mathcal{F}$ is a ψ-contraction. Then the following inhomogeneous equation:*

$$x=F(x)+v,$$

has a unique solution $x=R_F(v)$ *for every (fixed)* $v\in C(X)$. *Moreover,* $R_F(v)=K_F$ *if and only if* $v\leq K_F$, *where* K_F *is a unique fixed point of* F, *in other words,* $R_F^{-1}(K_F)$ *is a principal ideal of* $C(X)$.

Proof. Obviously $R_F(v)$ satisfies the equation considered. The constant mapping $c(x)\equiv v$ satisfies the condition $\psi_0\equiv 0$ and therefore $G=F+c$ is a ψ-contraction by Proposition 2. This yields the uniqueness of the solution. This implies that $R_F(v)=K_F$ if $v\leq K_F$. The converse is trivial. □

By the above theorem, the operator R_F can be regarded as the resolvent $(Id-F)^{-1}$.

Theorem 2 (Alternative of Fredholm). *Suppose that* $F\in\mathcal{F}$ *is regular. Then the following statements* (a) *and* (b) *are equivalent*:

(a) *there exists a unique solution of* $x=F(x)+v$ *for every* $v\in C(X)$;

(b) F *has a unique fixed point* K_F.

Proof. It suffices to show (a) assuming (b). Suppose, on the contrary, that there are two distinct solutions u, w for some $v\in C(X)$. (Note that the equation has at least one solution since F is regular.) From (b), it follows $u\cdot w\neq\phi$. Without loss of generality, we can assume $z\equiv$closure of $u-u\cdot w\neq\phi$. Then $u\cdot w\geq K_F$ since $u\cdot w\geq F(u)\cdot F(w)\geq F(u\cdot w)$. We now show $F(z)\geq z$. For otherwise, there exists a point $p\in u-u\cdot w$ such that $p\notin F(z)$. Thus, $p\notin F(z)+F(u\cdot w)+v=F(u)+v=u$, contrary to $p\in u$. Since $F(z)\geq z$ and $u\geq z$, $F^n(u)\geq z$ for any $n\geq 0$. Hence $K_F\geq z\geq u-u\cdot w$ and this contradiction completes the proof. □

Concerning fixed points of F, we have:

Theorem 3. *If* $R_F(x)\in C(X)$, *then* $L_F(x)$ *is a fixed point of* F.

Corollary. *Suppose that* $F\in\mathcal{F}$ *is regular and that* F *has a unique fixed point* K_F. *Then* $\limsup_{n\to\infty} F^n(x)=K_F$ *for every* $x\in C(X)$.

Before proving the theorem, we need:

Lemma. *Let* $F\in\mathcal{F}$ *and* $x\in C(X)$. *For any* $q\in F(x)$, *there exists at least one point* $p\in x$ *such that* $q\in F(\{p\})$.

Proof. For any $\varepsilon>0$, x can be represented as a finite sum $\sum_{j=1}^m x_j$ such that $x_j\in C(X)$ and $\operatorname{diam}(x_j)<\varepsilon$. Since $F(x)=\sum_{j=1}^m F(x_j)$, there exists x_i satisfying $q\in F(x_i)$. Continuing in this way, we find a sequence $x\geq y_1\geq y_2\geq\cdots$ such that $\operatorname{diam}(y_n)<2^{-n}$ and $q\in F(y_n)$. Let $p=\lim_{n\to\infty} y_n\in x$. Since $y_n\to\{p\}$ $(n\to\infty)$ in $C(X)$, we have $F(y_n)\to F(\{p\})$ $(n\to\infty)$ by the continuity of F. Hence $q\in F(\{p\})$. □

Proof of Theorem 3. Put $Q=L_F(x)\in C(X)$ for brevity. We first show $F(Q)\geq Q$. For any $p\in Q$, there exists a sequence $\{q_n\}\leq X$ such that $q_n\in F^{m_n}(x)$ and $q_n\to p$ $(n\to\infty)$ in X. By Lemma, there exists $r_n\in F^{m_n-1}(x)$ satisfying $q_n\in F(\{r_n\})$. Since $r_n\in R_F(x)$, without loss of generality, we can assume that $r_n\to r^*\in Q$ $(n\to\infty)$ in X. Therefore $\{r_n\}\to\{r^*\}$ $(n\to\infty)$ in $C(X)$ and we get $F(\{r_n\})\to F(\{r^*\})$. Hence $p\in F(\{r^*\})\leq F(Q)$.

We next show the converse inequality $F(Q)\leq Q$. Let $\{q_n\}$ be the same sequence in X as above. Since $F(\{q_n\})\leq F^{m_n+1}(x)$ and $F(\{q_n\})\to F(\{p\})$ $(n\to\infty)$

in $C(X)$, we get $F(\{p\}) \leq Q$. For any $\varepsilon > 0$, there exists $\delta_p > 0$ such that $F(N_p) \leq N_\varepsilon(F(\{p\}))$ where N_p = closure of $N_{\delta_p}(\{p\}) \cdot Q \in C(X)$. Since $\{N_{\delta_p}(\{p\})\}_{p \in Q}$ is an open covering of Q, there exist $p_1, \cdots, p_m \in Q$ such that $Q \leq \sum_{j=1}^m N_{p_j}$. Therefore

$$F(Q) \leq \sum_{j=1}^m F(N_{p_j}) \leq \sum_{j=1}^m N_\varepsilon(F(\{p_j\})) \leq N_\varepsilon(Q).$$

Since ε is arbitrary, we have $F(Q) \leq Q$. This completes the proof. □

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