3. *On Topological Dynamical Systems with Discrete Spectrum*

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§1. **Main results.** Throughout this note (X, T) is a topological dynamical system, i.e., a pair of a compact Hausdorff space X and a continuous map T of X to itself. Let $C(X)$ be the Banach space of all continuous complex functions on X with the usual supremum norm. By $E(X, T)$ [resp. $\sigma(X, T)$] we denote the set of all eigenfunctions [resp. eigenvalues] of U_T defined by $U_T(f)=f\circ T(f\in C(X))$. We say that (X, T) has *discrete spectrum* if the norm closed linear span of $E(X, T)$ is identical with $C(X)$. For any fixed $x\in X$ we put $O_T(x)=\{T^n x;\ n\in \boldsymbol{N}\}$ and $O_T^+(x)=\{T^n x;\ n\in \boldsymbol{Z}^+\}$, where $\boldsymbol{N}[\boldsymbol{Z}^+]$ is the set of all nonnegative [positive] integers. (X, T) is said to be *topologically transitive* if there exists some $p\in X$ for which $O_T(p)$ is dense in X. We distinguish the topological transitivity for (X, T) into the following two cases:

(A) There exists some $p\in X$ for which $O_T^+(p)$ is dense in X.

(B) There exists some $p\in X$ for which $O_T(p)$ is dense in X, and $O_T^+(x)$ is not dense in X for all $x\in X$.

The purpose of this paper is to clarify the structure of topologically transitive (X, T) with discrete spectrum. We say that (X, T) is *topologically conjugate* to a topological dynamical system (Y, S), in symbol $(X, T)\cong(Y, S)$, if there exists a homeomorphism ϕ of X onto Y such that $\phi\circ T=S\circ\phi$. Let $(X, T)\cong(Y, S)$. Then $\sigma(X, T)=\sigma(Y, S)$, (X, T) has discrete spectrum if and only if so has (Y, S), and further (X, T) satisfies (A) [(B)] if and only if (Y, S) satisfies (A) [(B)].

Let G be a compact abelian semigroup, $a\in G$ and L_a the translation on G defined by a. Then we get a topological dynamical system (G, L_a). Let $G_e=G\cup\{e\}$ be the adjunction of an identity e to G. This is also a compact abelian semigroup in which e is an isolated point. A semicharacter of G is a continuous function χ on G such that $\chi(g)\neq 0$ for some $g\in G$ and $\chi(st)=\chi(s)\chi(t)$ for all s, t in G. By $\hat{G}$ we denote the set of all semicharacters of G. G is said to be *separative* if for any distinct $s, t\in G$ there exists $\chi\in\hat{G}$ with $\chi(s)\neq\chi(t)$. As seen easily G_e is separative if and only if so is G. Further if G is separative, then the norm closed linear span of $\hat{G}$ is identical with $C(G)$. If there exists some $a\in G$ such that $\{a^n;\ n\in\boldsymbol{Z}^+\}$ is dense in G, then G is called a monotetic semigroup with the generator a. Under the above notations and terminology our main results are stated as follows.

Theorem 1. *(X, T) has discrete spectrum and satisfies the condition*

(A) *if and only if there exists a compact monothetic group G such that* $(X, T)\cong(G, L_a)$, *where a is the generator of G.*

Theorem 2. (X, T) *has discrete spectrum and satisfies the condition* (B) *if and only if there exists a separative compact monothetic semigroup G such that* $(X, T)\cong(G_e, L_a)$, *where a is the generator of G.*

The above theorems are generalizations of Halmos and von Neumann [1, Theorem 6] (cf. [4, Theorem 5.18]).

Remarks. (1) On the structure of compact monothetic semigroups it is investigated in detail by E. Hewitt [2]. We conclude from [2, p. 456] that a compact monothetic semigroup is separative if and only if it is of type I or type III in the sense of [2, Main theorem].

(2) If (X, T) has discrete spectrum and satisfies (A), then it follows from Theorem 1 that T must be a homeomorphism.

(3) If (X, T) has discrete spectrum and satisfies (B), then we see from Theorem 2 that the point p as in (B) is unique and $TX=X\setminus\{p\}$.

§2. Sketch of proof. Let G be a separative compact monothetic semigroup with a generator a and G_e the adjunction of the identity e to G. Then (G_e, L_a) satisfies (B) for $T=L_a$ and $p=e$. Further we see easily that a function $f\in C(G_e)$ is in $E(G_e, L_a)$ if and only if it is given in the form $f=c\chi$, where c is a nonzero constant and $\chi\in\hat{G}_e$. Since G_e is also separative, it follows from the Stone-Weierstrass theorem that (G_e, L_a) has discrete spectrum. Thus the "if" part of Theorem 2 is proved. Similarly the "if" part of Theorem 1 is shown.

Conversely suppose that (X, T) has discrete spectrum and $O_T(p)$ is dense in X for some $p\in X$. Let $\mathfrak{U}$ be the uniformity of X which induces the original topology of X. Then by the same way as in [3, Theorem 1], the family $\{T^n; n\in N\}$ of iterations of T is equicontinuous, i.e., for any index $\alpha\in\mathfrak{U}$ there corresponds to $\beta\in\mathfrak{U}$ such that $T^n\beta\subset\alpha$ for all $n\in N$. Let us define a map $*$ of $O_T(p)\times O_T(p)$ to $O_T(p)$ by $T^mp*T^np=T^{m+n}p$ $(m, n\in N)$. Then under the multiplication $*$, $O_T(p)$ becomes an abelian semigroup with the identity p. For any $\alpha\in\mathfrak{U}$ we take β, γ in $\mathfrak{U}$ such that $\beta\circ\beta\subset\alpha$ and $T^n\gamma\subset\beta$ for all $n\in N$. If $(T^kp, T^lp)\in\gamma$ and $(T^mp, T^np)\in\gamma$, then we have $(T^{k+m}p, T^{l+m}p)\in\beta$, $(T^{l+m}p, T^{l+n}p)\in\beta$, and hence $(T^{k+m}p, T^{l+n}p)=(T^kp*T^mp, T^lp*T^np)\in\alpha$. So that the multiplication $*$ on $O_T(p)$ is uniformly continuous and can be extended uniquely to a continuous map $*$ of $X\times X$ to X. Hence X is regarded as a compact abelian semigroup, denoted by G_T, under the multiplication $*$. Putting $a^n=T^np$ $(n\in Z^+)$ and $e=a^0=p$, we have $T(a^n)=T^{n+1}p=a*a^n(n\in N)$. This shows that T is the translation L_a on G_T. Therefore $(X, T)\cong(G_T, L_a)$. Since (X, T) has discrete spectrum, it follows that (G_T, L_a) has also discrete spectrum and G_T is separative. Let G be the closure of $\{a^n; n\in Z^+\}$. Then it is a separative compact monothetic subsemigroup of G_T with the generator a. If (X, T) satisfies (A), then $G_T=G$ and G becomes a compact monothetic group, because any compact monothetic semigroup with identity is a topological group (cf. [2]). On the other hand if (X, T) satisfies (B), then $G_T = G \cup \{e\}$. Consequently we get the "only if" parts of Theorems 1 and 2.

§ 3. **Conjugacy theorem.** Let (X, T), G and $a \in G$ be as in Theorem 1 [Theorem 2]. Then we have $\sigma(X, T) = \{\chi(a)\,;\, \chi \in \hat{G}\}[\sigma(X, T) = \{0\} \cup \{\chi(a)\,;\, \chi \in \hat{G}\}]$. On the other hand let G_1 and G_2 be separative compact monothetic semigroups with generators a_1 and a_2 respectively. Then we see from the discussion in [2, p. 456] that G_1 and G_2 are isomorphic if $\{\chi(a_1)\,;\, \chi \in \hat{G}_1\} = \{\chi(a_2)\,;\, \chi \in \hat{G}_2\}$. Accordingly from Theorems 1 and 2 we obtain the following conjugacy theorem, which is an analogue of Theorem 5.19 in [4].

Theorem 3. *Let (X, T) and (Y, S) be topologically transitive topological dynamical systems with discrete spectrum. Then $(X, T) \cong (Y, S)$ if and only if $\sigma(X, T) = \sigma(Y, S)$.*

References

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