19. *Infinitely Many Periodic Solutions for the Equation:* $u_{tt}-u_{xx}\pm|u|^{s-1}u=f(x,t)$

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1985)

1. Introduction. In this article we shall study the nonlinear wave equation:

(1)$_\pm$ $\quad v_{tt}-v_{xx}\pm|v|^{s-1}v=f(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

(2) $\quad v(0,t)=v(\pi,t)=0, \qquad t\in\boldsymbol{R},$

(3) $\quad v(x,t+2\pi)=v(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

where $s>1$ is a constant and $f(x,t)$ is a 2π-periodic function of t.

Our main result is as follows:

Theorem. *Assume that* $1<s<1+\sqrt{2}$ *and* $f(x,t)\in L^q_{\mathrm{loc}}([0,\pi]\times\boldsymbol{R})$ $(q=1/s+1)$ *is a* 2π*-periodic function of* t. *Then* (1)$_\pm$–(3) *possessess an unbounded sequence of weak solutions in* $L^{s+1}_{\mathrm{loc}}([0,\pi]\times\boldsymbol{R})$.

To prove our theorem, we convert the problem to a simpler one by a Legendre transformation which is used in H. Brézis, J. M. Coron and L. Nirenberg [2], that is, we use the dual variational formulation for (1)$_\pm$–(3). Next we use a perturbation result of P. H. Rabinowitz [3] asserting the existence of infinitely many critical points of perturbed symmetric functionals.

After completing this work, the author knew announcement of the result of J. P. Ollivry [6]. His result is analogous to ours for (1)$_+$–(3) but under the following conditions:

$$1<s<2 \quad \text{and} \quad f(x,t)\in E \quad (\text{see } (4)).$$

Our result obviously contains his result. Moreover our growth restriction $1<s<1+\sqrt{2}$ coincides with the condition which ensures the existence of an unbounded sequence of solutions of the semilinear elliptic equation:

$$-\Delta u=|u|^{s-1}u+f(x), \qquad x\in\Omega,$$
$$u=0, \qquad x\in\partial\Omega,$$

where $\Omega\subset\boldsymbol{R}^2$ is a smooth bounded domain (see P. H. Rabinowitz [3]).

2. Outline of the proof of Theorem. We shall only give outline of proof. Details will be published elsewhere.

We shall deal with the case (1)$_+$–(3) (the argument is essentially the same for the case (1)$_-$–(3)).

Let $\Omega=(0,\pi)\times(0,2\pi)$.

We shall consider the operator $Au=u_{tt}-u_{xx}$ acting on functions in $L^1(\Omega)$ satisfying (2), (3). Denote by N the kernel of A. Consider the space

(4) $$E=\left\{u\in L^q(\Omega)\,;\int_\Omega u\phi=0 \text{ for all } \phi\in N\cap L^{s+1}(\Omega)\right\}$$

with L^q norm $\|\cdot\|_q$.

For any $u \in E$ there exists a unique $Ku \in C^{0,\alpha}(\bar{\Omega}) \cap E$ with $\alpha = 1 - 1/q$ such that $A(Ku) = u$. The operator $K: E \to E^*$ is compact.

For a given $f \in L^q(\Omega)$ we define the functional $I(u) \in C^1(E, \boldsymbol{R})$ by

$$I(u) = -(1/2)(-Ku, u) + (1/q)\|u - f\|_q^q$$

where $(\cdot, \cdot)$ denotes the duality product between E^* and E. There is one-to-one correspondence between the critical points of $I(u)$ and the weak solutions of $(1)_+$–(3). This is so-called dual variational formulation of the problem $(1)_+$–(3).

For technical reasons, we shall replace $I(u)$ by a modified functional $J(u) \in C^1(E, \boldsymbol{R})$ defined by

$$J(u) = -(1/2)(-Ku, u) + (1/q)\|u\|_q^q + (1/q)\cdot\psi(u)\cdot(\|u - f\|_q^q - \|u\|_q^q),$$

where $\psi(u)$ will be defined analogously as in P. H. Rabinowitz [3].

Then we have the following propositions.

Proposition 1. *There is a constant $\beta = \beta(\|f\|_q) > 0$ such that for $u \in E$,*

$$|J(u) - J(-u)| \le \beta\cdot(|J(u)|^{(q-1)/q} + 1).$$

Proposition 2. *There is a constant $M = M(\|f\|_q) > 0$ such that*

(i) *$J(u) \in C^1(E, \boldsymbol{R})$ satisfies Palais-Smale condition on*

$$\hat{A}_M = \{u \in E\,;\, J(u) \ge M\}.$$

(ii) *$J(u) \ge M$ and $J'(u) = 0$ imply that $I(u) = J(u)$ and $I'(u) = 0$.*

Note that K is a compact self-adjoint operator in $E \cap L^2(\Omega)$. Its eigenvalues are $\{1/(j^2 - k^2)\,;\, j \ne k\}$. We rearrange the eigenvalues in the following order, denoted by

$$-\mu_1 \le -\mu_2 \le -\mu_3 \le \cdots < 0 < \cdots \le \nu_3 \le \nu_2 \le \nu_1$$

with repetitions according to the multiplicity of each eigenvalue and denote by e_j and f_j the eigenfunctions which are corresponding to $-\mu_j$ and ν_j respectively. We assume moreover $\|e_j\|_q = \|f_j\|_q = 1$ for all $j \in \boldsymbol{N}$. Next we shall define the spaces E_n, $E_n^\perp$ by

$$E_n = \operatorname{span}\{e_1, e_2, \cdots, e_n\},$$

$$E_n^\perp = \{u \in E\,;\, (e_i, u) = 0 \text{ for } i = 1, 2, \cdots, n\}.$$

Proposition 3. *There are constants $a_n > 0$ such that*

$$(-Ku, u) \le a_n\cdot\|u\|_q^2 \qquad \text{for all } u \in E_n^\perp.$$

Moreover for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$a_n \le C_\delta\cdot n^{-2(q-1)/q+\delta} \qquad \text{for all } n \in \boldsymbol{N}.$$

Clearly there is a sequence of numbers: $0 < R_1 < R_2 < \cdots$ such that

$$J(u) \le 0 \qquad \text{for all } u \in E_n \text{ with } \|u\|_q \ge R_n.$$

Let $B_R = \{u \in E\,;\, \|u\|_q \le R\}$ and $D_n = B_{R_n} \cap E_n$. Set

$$\Gamma_n = \{h \in C(D_n, E)\,;\, h \text{ is odd and } h(u) = u \text{ if } \|u\|_q = R_n\}.$$

Define

$$b_n = \inf_{h \in \Gamma_n} \sup_{u \in D_n} J(h(u)) \qquad \text{for } n \in \boldsymbol{N}.$$

Using the Borsuk-Ulam theorem and Proposition 3, we have

Proposition 4. *For every $\delta > 0$ there is a constant $C_\delta > 0$ such that*

$$b_n \ge C_\delta\cdot n^{2(q-1)/(2-q)-\delta} \qquad \text{for all } n \in \boldsymbol{N}. \tag{5}$$

Let

$U_n=\{u=t\cdot e_{n+1}+w\,;\ t\in[0,R_{n+1}],\ w\in B_{R_{n+1}}\cap E_n \text{ and } \|u\|_q\le R_{n+1}\}.$
$\Lambda_n=\{H\in C(U_n,E)\,;\ H|_{D_n}\in\Gamma_n,\ H(u)=u \text{ if } \|u\|_q=R_{n+1}$
$\text{or } u\in(B_{R_{n+1}}\backslash B_{R_n})\cap E_n\}.$

Define

$$c_n=\inf_{H\in\Lambda_n}\sup_{u\in U_n} J(H(u)) \qquad \text{for } n\in \mathbf{N}.$$

P. H. Rabinowitz [3] proved the following perturbation result.

Proposition 5. *Assume that $c_n>b_n\ge M$. Then $J(u)$ possesses a critical value in $[c_n,\infty)$.*

Hence to prove our theorem, it suffices to show that $c_n=b_n$ is not possible for all large n. We have the following:

Proposition 6. *If $c_n=b_n$ for all $n\ge n_0$, there exists a constant $\gamma=\gamma(n_0)$ such that*

$$b_n\le\gamma\cdot n^q \qquad \text{for all } n\in\mathbf{N}. \tag{6}$$

Thus comparing (5) and (6), we see the inequalities are incompatible if

$$\sqrt{2}<q<2, \quad \text{i.e.,} \quad 1<s<1+\sqrt{2}.$$

Hence there is a sequence $\{u_n\}_{n=1}^{\infty}$ of critical points of $I(u)$ such that

$$I(u_n)\to\infty \qquad \text{as } n\to\infty.$$

So there exists the sequence of solutions $\{v_n\}_{n=1}^{\infty}$ of $(1)_+$–(3) corresponding to u_n satisfying

$$\|v_n\|_{s+1}\to\infty \qquad \text{as } n\to\infty.$$

3. Remarks.

Remark 1. K. Tanaka [5] proved for all $s\in(1,\infty)$ there is a dense set of $f\in L^2(\Omega)$ for which $(1)_+$–(3) possesses a weak solution. This result holds for more general equation:

$$u_{tt}-u_{xx}+g(u)=f(x,t), \qquad (x,t)\in(0,\pi)\times\mathbf{R}, \tag{7}$$

$$u(0,t)=u(\pi,t)=0, \qquad t\in\mathbf{R}, \tag{8}$$

$$u(x,t+T)=u(x,t), \qquad (x,t)\in(0,\pi)\times\mathbf{R}, \tag{9}$$

where $g(s)$ is a continuous function and $f(x,t)$ is a T-periodic function. (We don't assume the monotonicity of $g(s)$ and $T/\pi\in\mathbf{Q}$.)

Theorem. *Assume that there exist constants $C_1>0$ and $C_2>0$ such that*

$$\int_0^s g(\tau)d\tau\le C_1\cdot sg(s)+C_2 \qquad \text{for all } s\in\mathbf{R},$$

$$\lim_{|s|\to\infty}\frac{g(s)}{s}=\infty.$$

Then for all $f(x,t)$ in a dense subset Ξ of L^2, (7)–(9) possesses a weak solution (or equivalently the range of the operator: $u\to u_{tt}-u_{xx}+g(u)$ is dense in L^2).

Remark 2. Using Proposition 3, we can give a simple proof of the result of P. H. Rabinowitz [4] by the dual variational method.

Acknowledgement. The author would like to thank Prof. Haruo Sunouchi for his advice and encouragement.

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