18. Well-posedness of the Cauchy Problem for Some Weakly Hyperbolic Operators in Gevrey Classes

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§0. **Introduction.** We consider whether we can determine a function space in which the Cauchy problem for a given weakly hyperbolic operator is well-posed or not.

This question has been studied by several mathematicians.

The results independent of the lower order terms were obtained by Ohya [4] and Bronstein [1] etc., which show that the multiplicity of the characteristic roots determines the well-posed class.

On the other hand, in [3] Ivrii presented two interesting examples.

(I) Let $P=\partial_t^2-t^{2\mu}\partial_x^2+at^\nu\partial_x$, where μ and ν are non-negative integers and $a\neq 0$. When $0\leqq\nu<\mu-1$, the Cauchy problem for P is $\gamma_{\mathrm{loc}}^{(\kappa)}$-well-posed if and only if $1\leqq\kappa<(2\mu-\nu)/(\mu-\nu-1)$.

(II) Let $P=\partial_t^2-x^{2\mu}\partial_x^2+ax^\nu\partial_x$, where μ,ν and a are the same as (I). When $0\leqq\nu<\mu$, the Cauchy problem for P is $\gamma_{\mathrm{loc}}^{(\kappa)}$-well-posed if and only if $1\leqq\kappa<(2\mu-\nu)/(\mu-\nu)$.

These two examples show that the lower order terms have a great effect on the well-posed class.

Igari [2], Uryu [6] and Uryu-Itoh [7] extended Ivrii's examples for more general operators respectively.

In this paper we shall extend (II) to some weakly hyperbolic operators of order m and of variable multiplicity.

§1. **Statement of results and remarks.**

Definition 1 ($\gamma_{\mathrm{loc}}^{(\kappa)}, \gamma^{(\kappa)}; \kappa\geqq 1$). $f(x)\in\gamma_{\mathrm{loc}}^{(\kappa)}$ implies that $f(x)\in C^\infty(\mathbf{R}^n)$ and for any compact set $K\subset\mathbf{R}^n$ there exist constants $c, R>0$ such that $|D_x^\alpha f(x)|\leqq cR^{|\alpha|}|\alpha|!^\kappa$, $x\in K$, for any α. $f(x)\in\gamma^{(\kappa)}$ implies that this estimate holds for any $x\in\mathbf{R}^n$.

Let L be

$$L=L_0(t,x,D_t,D_x)+L_1(t,x,D_t,D_x), \tag{1}$$

where

$$L_0(t,x,D_t,D_x)=D_t^m+\textstyle\sum_{k=1}^m \sigma(x)^{k\mu}\left(\sum_{|\alpha|=k} a_{k\alpha}(t,x)D_x^\alpha\right)D_t^{m-k}$$

and

$$L_1(t,x,D_t,D_x)=\textstyle\sum_{k=1}^m\sum_{j=1}^k \sigma(x)^{\nu k-j}\left(\sum_{|\alpha|=k-j} b_{jk\alpha}(t,x)D_x^\alpha\right)D_t^{m-k}.$$

We assume the following conditions on L.

(A-1) τ-roots of $\tau^m+\sum_{k=1}^m\sum_{|\alpha|=k} a_{k\alpha}(t,x)\xi^\alpha\tau^{m-k}=0$ are real and distinct.

(A-2) $a_{k\alpha}(t,x),\ b_{jk\alpha}(t,x)\in\mathcal{B}([0,T],\gamma^{(\kappa)})$.

(A-3) $\sigma(x)\in\gamma^{(\kappa)}$ and is a real-valued function.

(A-4) $\mu, \nu_0, \cdots, \nu_{m-1}$ are non-negative integers such that $\mu \geqq 1$ and $0 \leqq \nu_j \leqq j\mu$ $(j=0, \cdots, m-1)$.

Now we shall define important numbers $\nu(i)$, $\rho(i)$ and ρ. For $i=1, \cdots, m-1$, $\nu(i)=\nu_i/(i\mu)$ and $\rho(i)=1+i\{1-\nu(i)\}$. And $\rho=\max\{\rho(1), \cdots, \rho(m-1)\}$.

Then we have the following theorem.

Theorem 1. *Under* (A-1)–(A-4), *if* $1 \leqq \kappa < \rho/(\rho-1)$, *the Cauchy problem for* L:

$$(CP) \begin{cases} Lu(t,x)=f(t,x) & \textit{in } (0,T]\times \mathbf{R}^n \\ D_t^i u(t,x)|_{t=0}=u^i(x), \quad i=0, \cdots, m-1 & \textit{on } \mathbf{R}^n \end{cases}$$

is $\gamma_{loc}^{(\kappa)}$*-well-posed,* i.e. *for any* $u^i(x) \in \gamma_{loc}^{(\kappa)}$ $(i=0, \cdots, m-1)$ *and any* $f(t,x) \in \mathcal{B}([0,T], \gamma_{loc}^{(\kappa)})$ *there exists a unique solution* $u(t,x) \in \mathcal{B}([0,T], \gamma_{loc}^{(\kappa)})$ *of* (CP).

Remark 1. When $\rho=1$, (CP) is C^∞-well-posed (see [5]).

Remark 2. In the case of finite degeneracy, our sufficient condition is best (see [3]).

§2. Sketch of the proof of Theorem 1. We shall reduce Theorem 1 to Theorem 2.

Definition 2. We say that $f(x) \in H^\infty$ belongs to $\Gamma^{(\kappa)}$ if there exist constants c, $R>0$ such that $\|D_x f(x)\| \leqq cR^{|\alpha|}|\alpha|!^\kappa$ for any α, where $\|\cdot\|$ denotes L^2-norm with respect to x.

Definition 3 (cf. [7]). We say that a symbol $h(x,\xi)$ belongs to $S^m(\kappa)$ if there exist constants c_α, $R>0$ such that for any α, β

$$|\partial_\xi^\alpha D_x^\beta h(x,\xi)| \leqq c_\alpha R^{|\beta|}|\beta|!^\kappa \langle\xi\rangle^{m-|\alpha|}, \qquad (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Let P be a pseudo-differential operator

$$(2) \qquad P=P(t,x,D_t,D_x)=P_0(t,x,D_t,D_x)+P_1(t,x,D_t,D_x).$$

$P_0(t,x,\tau,\xi)=\prod_{j=1}^m \{\tau-\sigma(x)^\mu \lambda_j(t,x,\xi)\}$, where $\lambda_j(t,x,\xi) \in \mathcal{B}([0,T], S^1(\kappa))$ are real-valued and $|(\lambda_i-\lambda_j)(t,x,\xi)| \geqq \delta\langle\xi\rangle$ for some constant $\delta>0$ if $i \neq j$. Further

$$P_1(t,x,\tau,\xi)=\sum_{k=1}^m \sum_{j=1}^k \sigma(x)^{\nu_k - j} b_{k-j}(t,x,\xi)\tau^{m-k},$$

where

$$b_j(t,x,\xi) \in \mathcal{B}([0,T], S^j(\kappa)).$$

Then we get the following theorem.

Theorem 2. *Under* (A-1)–(A-4), *if* $1 \leqq \kappa < \rho/(\rho-1)$, *the Cauchy problem for* P *is* $\Gamma^{(\kappa)}$*-well-posed.*

In order to prove Theorem 1, it is sufficient to show Theorem 2. For since an operator (1) is changed into another operator (2) by space like transformation, we can see that a domain of dependence is finite. Hence using a partition of unity, Theorem 1 follows from Theorem 2.

§3. Sketch of the proof of Theorem 2. We shall prove Theorem 2 by the method of successive approximation. Therefore we decompose P as follows and consider the following scheme.

$$P(t,x,D_t,D_x)=Q_0(t,x,D_t,D_x)+Q_1(t,x,D_t,D_x),$$

where

$$Q_0(t,x,D_t,D_x)=P_0(t,x,D_t,D_x)+\sum_{k=1}^m b_0(t,x,D_x)D_t^{m-k}$$

and

$$Q_1(t, x, D_t, D_x)=\sum_{k=2}^{m}\sum_{j=1}^{k-1}\sigma(x)^{\nu_k-j}b_{k-j}(t, x, D_x)D_t^{m-k}.$$

$$(3)_0 \qquad \begin{cases} Q_0u_0(t, x)=f(t, x) & \text{in } (0, T]\times\mathbf{R}^n \\ D_t^iu_0(t, x)|_{t=0}=u^i(x), \quad i=0, \cdots, m-1 & \text{on } \mathbf{R}^n \end{cases}$$

and for $j\geqq 1$

$$(3)_j \qquad \begin{cases} Q_0u_j(t, x)=-Q_1u_{j-1}(t, x) & \text{in } (0, T]\times\mathbf{R}^n \\ D_t^iu_j(t, x)|_{t=0}=0, \quad i=0, \cdots, m-1 & \text{on } \mathbf{R}^n. \end{cases}$$

Since the Cauchy problem for Q_0 is H^∞-well-posed (see [5]), it is sufficient to show that the formal solution

$$u(t, x)=\sum_{j=0}^{\infty}u_j(t, x) \text{ converges in } \mathcal{B}([0, T], \Gamma^{(\kappa)}).$$

For this purpose we consider the following Cauchy problem.

$$(4) \qquad \begin{cases} Q_0v(t, x)=g(t, x) \\ D_t^iv(t, x)|_{t=0}=0, \quad i=0, \cdots, m-1 \end{cases}$$

where $g(t, x)\in\mathcal{B}([0, T], \Gamma^{(\kappa)})$ such that for any fixed integer $s\geqq 1$ $D_t^ig|_{t=0}=0$, $0\leqq i\leqq s-1$. We may assume that for any $r\geqq 0$ there exist constants c, R, $M>0$ such that $\|\Lambda^rg(t, x)\|\leqq cR^rr!^\kappa t^se^{Mrt}$. For simplicity we use the notation $w_r(s, t, R)=R^rr!^\kappa t^se^{Mrt}$.

We assume the existence of solutions of (4).

Lemma 1. *Let* $\Phi_r(t)=\sum_{k=0}^{m-1}(r+1)^{m-(k+1)}\sum_{i+j=k}\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^jv\|$. *Thus for any* $r\geqq 0$ *there exists a constant* $A>0$ *such that for sufficiently large* R, M, s $\Phi_r(t)\leqq cAs^{-1}w_r(s, t, R)$.

The following lemmas follow from Lemma 1.

We note that $\nu_i=0$ or there exist non-negative integers p_i such that $p_i\mu<\nu_i\leqq(p_i+1)\mu$, $i=1, \cdots, m-1$.

Lemma 2. *For any* $r\geqq 0$, *the following estimate holds.*

$$\begin{aligned}\|\Lambda^rQ_1v\|\leqq c'cA\sum_{i=1}^{m-1}&[s^{-\rho(i)}\{(r+i)\cdots(r+1)\}^{-\nu(i)\kappa} \\ &+s^{-(q+i-p_i)}\{(r+i)\cdots(r+i-q+1)\}^{1-\kappa}\{(r+i-q)\cdots(r+i-p_i)\}^{-\kappa}] \\ &\times w_r(s+\rho(i)-1, t, R),\end{aligned}$$

where $c'>0$ *and* q *is a positive integer such that* $p_i+1-q\geqq 0$.

Lemma 3. *The Cauchy problem for* Q_0 *is* $\Gamma^{(\kappa)}$*-well-posed.*

Lemma 4. *For any fixed integer* $s\geqq 1$ *there exists* $N=N(s)\in\mathbf{N}$ *such that for any* $j\geqq N-1$ $D_t^iu_j|_{t=0}=0$, $0\leqq i\leqq s+m-3$.

Therefore we may assume that for any $r\geqq 0$

$$(5) \qquad \|\Lambda^rQ_1u_{N-1}\|\leqq cw_r(s, t, R).$$

Lemma 5. *Under* (5), *if* $1\leqq\kappa<\rho/(\rho-1)$, *there exist constants* $A', B, \gamma>0$ *which are independent of* r *such that*

$$(6) \qquad \|\Lambda^ru_{N+n}\|\leqq cA'B^nn^{-\gamma n}w_r(s, t, 2^\kappa R) \qquad \text{for } n=0, 1, 2, \cdots.$$

From Lemma 3 and Lemma 5 we find that if $1\leqq\kappa<\rho/(\rho-1)$, the formal solution converges in $\mathcal{B}([0, T], \Gamma^{(\kappa)})$. Hence we obtain the existence of solutions. And if we set $f(t, x)=0$ and $u^i(x)=0$ $(i=0, \cdots, m-1)$, then we can get the inequality similar to (6) for $u(t, x)$. Therefore we obtain the uniqueness of solutions.

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References

[1] M. D. Bronstein: The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity. Trans. Moscow Math. Soc., **41**, 87–103 (1982).

[2] K. Igari: An admissible data class of the Cauchy problem for non-strictly hyperbolic operators. J. Math. Kyoto Univ., **21**, 351–373 (1981).

[3] V. Ja. Ivrii: Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity. Sib. Mat. Zh., **17**, 921–931 (1976).

[4] Y. Ohya: Le problème de Cauchy pour les équations hyperboliques à caractéristique multiple. J. Math. Soc. Japan, **16**, 268–286 (1964).

[5] H. Uryu: The Cauchy problem for weakly hyperbolic equations (II); Infinite degenerate case. Tokyo J. Math., **3**, 99–113 (1980).

[6] ——: Conditions for well-posedness in Gevrey classes of the Cauchy problems for Fuchsian hyperbolic operators (to appear in Publ. RIMS, Kyoto Univ.).

[7] H. Uryu and S. Itoh: Well-posedness in Gevrey classes of the Cauchy problems for some second order weakly hyperbolic operators (to appear in Funkcial Ekvac.).