17. *Cauchy Problems for Fuchsian Hyperbolic Equations in Spaces of Functions of Gevrey Classes*

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1985)

In this paper, we deal with the Cauchy problem for Fuchsian hyperbolic equations with Gevrey coefficients, and establish the well posedness of the problem in spaces of functions of Gevrey classes.

1. Problem. Let us consider the Cauchy problem:

$$(\mathrm{P})\qquad \begin{cases} t^k\partial_t^m u+\sum_{\substack{j+|\alpha|\le m\\ j<m}} t^{p(j,\alpha)}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha u=f(t,x),\\ \partial_t^i u|_{t=0}=u_i(x), \qquad i=0,1,\cdots,m-k-1,\end{cases}$$

where $(t,x)=(t,x_1,\cdots,x_n)\in[0,T]\times\boldsymbol{R}^n$ $(T>0)$, $m\in\boldsymbol{N}$ $(=\{1,2,\cdots\})$, $k\in\boldsymbol{Z}_+$ $(=\{0,1,2,\cdots\})$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\boldsymbol{Z}_+^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $p(j,\alpha)\in\boldsymbol{Z}_+$ $(j+|\alpha|\le m$ and $j<m)$, $a_{j,\alpha}(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$ $(j+|\alpha|\le m$ and $j<m)$, $\partial_t=\partial/\partial t$, and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. Assume the following condition:

(A-1) $0\le k\le m$.

(A-2) $p(j,\alpha)\in\boldsymbol{Z}_+$ $(j+|\alpha|\le m$ and $j<m)$ satisfy

$$\begin{cases} p(j,\alpha)=k+\langle\nu,\alpha\rangle, & \text{when } j+|\alpha|=m \text{ and } j<m,\\ p(j,\alpha)>k-m+j, & \text{when } j+|\alpha|<m \text{ and } |\alpha|>0,\\ p(j,\alpha)\ge k-m+j, & \text{when } j+|\alpha|<m \text{ and } |\alpha|=0\end{cases}$$

for some $\nu=(\nu_1,\cdots,\nu_n)\in\boldsymbol{Q}^n$ such that $\nu_i\ge 0$ $(i=1,\cdots,n)$, where $\langle\nu,\alpha\rangle=\nu_1\alpha_1+\cdots+\nu_n\alpha_n$.

(A-3) All the roots $\lambda_i\ (t,x,\xi)$ $(i=1,\cdots,m)$ of

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=0$$

are *real*, *simple* and *bounded* on $\{(t,x,\xi)\in[0,T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n\,;|\xi|=1\}$.

Then, the equation is one of the most fundamental examples of Fuchsian hyperbolic equations. The characteristic exponents $\rho=0,1,\cdots,m-k-1,\ \rho_1(x),\cdots,\rho_k(x)$ are defined by the roots of

$$0=\rho(\rho-1)\cdots(\rho-m+1)+a_{m-1}(x)\rho(\rho-1)\cdots(\rho-m+2)$$
$$+\cdots+a_{m-k}(x)\rho(\rho-1)\cdots(\rho-m+k+1),$$

where $a_j(x)=(t^{p(j,(0,\cdots,0))-k+m-j}a_{j,(0,\cdots,0)}(t,x))|_{t=0}$ $(j<m)$.

2. Well posedness in $C^\infty([0,T],\mathcal{E}(\boldsymbol{R}^n))$. Let $\mathcal{E}(\boldsymbol{R}^n)$ be the Schwartz space on $\boldsymbol{R}^n$ and let $C^\infty([0,T],\mathcal{E}(\boldsymbol{R}^n))$ be the space of all C^∞ functions on $[0,T]$ with values in $\mathcal{E}(\boldsymbol{R}^n)$. Then, by applying the result in Tahara [6] we have

Theorem 1. *Assume that* (A-1)~(A-3) *and the condition:*

(T) $p(j,\alpha)\ge k-m+j+\langle\nu,\alpha\rangle+|\alpha|$, *when* $j+|\alpha|<m$ *and* $|\alpha|>0$

hold, and that $\rho_1(x),\cdots,\rho_k(x)\notin\{\lambda\in\boldsymbol{Z}\,;\lambda\ge m-k\}$ *for any* $x\in\boldsymbol{R}^n$. *Then, for* *any* $f(t, x) \in C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ *and any* $u_i(x) \in \mathcal{E}(\boldsymbol{R}^n)$ $(i=0, 1, \cdots, m-k-1)$ *there exists a unique solution* $u(t, x) \in C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ *of* (P). *In addition, the solution has a finite propagation speed.*

In Theorem 1, the condition (T) seems to be essential to the well posedness in $C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$. In fact, when $\nu_1 = \cdots = \nu_n$ holds, the necessity of (T) is easily obtained from results in Mandai [4]. Therefore, if we want to consider the case without (T), we must treat the problem (P) in suitable subclasses of $C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$.

3. Well posedness in $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$. A function $f(x)$ $(\in C^\infty(\boldsymbol{R}^n))$ is said to belong to the *Gevrey class* $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$, if $f(x)$ satisfies the following; for any compact subset K of $\boldsymbol{R}^n$ there are $C>0$ and $h>0$ such that

$$\sup_{x \in K} |\partial_x^\alpha f(x)| \leqq C h^{|\alpha|} (|\alpha|!)^s \qquad \text{for any } \alpha \in \boldsymbol{Z}_+^n.$$

We denote by $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ the space of all C^∞ functions on $[0, T]$ with values in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ equipped with the topology in Komatsu [3]. In other words, $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ is the space of all functions $g(t, x) \in C^\infty([0, T] \times \boldsymbol{R}^n)$ satisfying the following; for any $i \in \boldsymbol{Z}_+$ and any compact subset K of $\boldsymbol{R}^n$ there are $C>0$ and $h>0$ such that

$$\sup_{[0,T]\times K} |\partial_t^i \partial_x^\alpha g(t, x)| \leqq C h^{|\alpha|} (|\alpha|!)^s \qquad \text{for any } \alpha \in \boldsymbol{Z}_+^n.$$

Now, let us consider the problem (P) in $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ under (A-1) $\sim$ (A-3). Let $p(j, \alpha)$ $(j+|\alpha|<m$ and $|\alpha|>0)$ and $\nu = (\nu_1, \cdots, \nu_n)$ be as in (A-2). Define the irregularity index σ $(\geqq 1)$ by

$$\sigma = \max\left[1, \max_{\substack{j+|\alpha|<m\\ |\alpha|>0}} \left\{\min_{\tau \in \mathfrak{S}_n} \left(\max_{1 \leqq r \leqq n} M_{j,\alpha}(\tau, r)\right)\right\}\right],$$

where $\mathfrak{S}_n$ is the permutation group of n-numbers and

$$M_{j,\alpha}(\tau, r) = \frac{\sum_{i=1}^{r} (\nu_{\tau(i)} - \nu_{\tau(r)})\alpha_{\tau(i)} + (m-j)\nu_{\tau(r)} - p(j, \alpha) + k}{(m-j-|\alpha|)(\nu_{\tau(r)}+1)}.$$

Impose the following conditions:

(A-4) $1 < s < \sigma/(\sigma-1)$.

(A-5) $a_{j,\alpha}(t, x) \in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ $(j+|\alpha| \leqq m$ and $j<m)$.

When $\sigma=1$, (A-4) is read $1<s<\infty$. Then, we have

Theorem 2. *Assume that* (A-1) $\sim$ (A-5) *hold and that* $\rho_1(x), \cdots, \rho_k(x) \notin \{\lambda \in \boldsymbol{Z};\ \lambda \geqq m-k\}$ *for any* $x \in \boldsymbol{R}^n$. *Then, for any* $f(t, x) \in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *and any* $u_i(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ $(i=0, 1, \cdots, m-k-1)$ *there exists a unique solution* $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *of* (P). *In addition, the solution has a finite propagation speed.*

Remark. (1) $\sigma=1$ is equivalent to (T).

(2) When $\nu_1 = \cdots = \nu_n$ $(=\nu_*)$, σ is given by

$$\sigma = \max\left\{1, \max_{\substack{j+|\alpha|<m\\ |\alpha|>0}} \left(\frac{(m-j)\nu_* - p(j, \alpha) + k}{(m-j-|\alpha|)(\nu_*+1)}\right)\right\}.$$

(3) When $\nu_1 = \cdots = \nu_n$ holds, the well posedness of (P) in $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ is obtained by Uryu [7]. But, even in this case, our condition (A-4) is better than his.

4. Examples. We give here some typical examples of the case $k=0$, that is, the non-characteristic case.

Example 1. Let P_1 be of the form

$$P_1=\partial_t^2-t^{2\nu}\partial_x^2+t^p a(t,x)\partial_x+b(t,x)\partial_t+c(t,x),$$

where $(t,x)\in[0,T]\times\boldsymbol{R}$ and $2\nu,\ p\in\boldsymbol{Z}_+$. Then, σ is given by

$$\sigma=\max\left\{1,\frac{2\nu-p}{\nu+1}\right\}.$$

Therefore, if $\nu-1>p$, we have $\sigma>1$ and (A-4) is given by $1<s<(2\nu-p)/(\nu-p-1)$. This coincides with the condition in examples by Ivrii [2], Igari [1] and Uryu [7].

Example 2. Let P_2 be of the form

$$P_2=\partial_t^2-t^{2\nu_1}\partial_{x_1}^2-t^{2\nu_2}\partial_{x_2}^2+t^{p_1}a_1(t,x)\partial_{x_1}+t^{p_2}a_2(t,x)\partial_{x_2}+b(t,x)\partial_t+c(t,x),$$

where $(t,x)\in[0,T]\times\boldsymbol{R}^2$ and $2\nu_1, 2\nu_2, p_1, p_2\in\boldsymbol{Z}_+$. Then, σ is given by

$$\sigma=\max\left\{1,\frac{2\nu_1-p_1}{\nu_1+1},\frac{2\nu_2-p_2}{\nu_2+1}\right\}.$$

Example 3. Let P_3 be of the form

$$P_3=\partial_t(\partial_t^2-t^{2\nu_1}\partial_{x_1}^2-t^{2\nu_2}\partial_{x_2}^2)+t^p a(t,x)\partial_{x_1}\partial_{x_2},$$

where $(t,x)\in[0,T]\times\boldsymbol{R}^2$ and $2\nu_1, 2\nu_2, p\in\boldsymbol{Z}_+$. Then, σ is given by

$$\sigma=\begin{cases}\max\left\{1,\dfrac{3\nu_1-p}{\nu_1+1},\dfrac{\nu_1+2\nu_2-p}{\nu_2+1}\right\}, & \text{when } 0\leqq\nu_1\leqq\nu_2,\\[2ex] \max\left\{1,\dfrac{2\nu_1+\nu_2-p}{\nu_1+1},\dfrac{3\nu_2-p}{\nu_2+1}\right\}, & \text{when } 0\leqq\nu_2\leqq\nu_1.\end{cases}$$

Details and proofs will be published elsewhere.

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