# 88. On Some Class Number Relations for Galois Extensions

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Let $k$ be an algebraic number field of finite degree over $\boldsymbol{Q}$, $k_v$ be the completion of $k$ at a place $v$. When $v$ is non-archimedean, we put $v=\mathfrak{p}$ and denote by $\mathfrak{o}_\mathfrak{p}$ the ring of $\mathfrak{p}$-adic integers in $k_\mathfrak{p}$. We denote by $S_\infty$ the set of all archimedean places of $k$.

Let $T$ be a torus defined over $k$. We denote by $T(k)$, $T(k_v)$ the subgroups of $T$ of points rational over $k$, $k_v$, respectively. The adele group of $T$ over $k$ will be written $T(k_A)$. The unique maximal compact subgroup of $T(k_\mathfrak{p})$ will be written $T(\mathfrak{o}_\mathfrak{p})$. The latter group is described as

$$T(\mathfrak{o}_\mathfrak{p})=\{x \in T(k_\mathfrak{p})\,;\,|\xi(x)|_\mathfrak{p}=1 \quad \text{for all } \xi \in \hat{T}(k_\mathfrak{p})\}$$

where $\hat{T}=\mathrm{Hom}\,(T, G_m)$, the character module of $T$ and $\hat{T}(k_\mathfrak{p})$ is the submodule of $\hat{T}$ of characters defined over $k_\mathfrak{p}$. Finally, we put

$$T(k_A)_\infty=\prod_{v\in S_\infty} T(k_v)\times \prod_{\mathfrak{p}} T(\mathfrak{o}_\mathfrak{p})$$

and define the class number of $T$ over $k$ by

$$h_T=[T(k_A): T(k)T(k_A)_\infty].$$

Let $K$ be a finite Galois extension of $k$. If $T=R_{K/k}(G_m)$, the torus obtained from the multiplicative group $G_m$ by the restriction of the field of definition from $K$ to $k$, then $h_T$ coincides with the usual class number $h_K$ of $K$. Consider the exact sequence of tori defined over $k$:

$$0\longrightarrow R^{(1)}_{K/k}(G_m)\longrightarrow R_{K/k}(G_m)\xrightarrow{N} G_m\longrightarrow 0$$

where $N$ is the norm map for $K/k$ and $R^{(1)}_{K/k}(G_m)=\mathrm{Ker}\,N$. As mentioned above, tori $G_m$, $R_{K/k}(G_m)$ have class numbers $h_k$, $h_K$, respectively. We shall denote by $h_{K/k}$ the class number of the torus $R^{(1)}_{K/k}(G_m)$. Then one obtains a positive rational number $E(K/k)$ invariantly attached to a Galois extension $K/k$:

$$E(K/k)\overset{\text{def}}{=}\frac{h_K}{h_k h_{K/k}}.$$

The celebrated formula of Gauss ($h^+_K=h^*_K 2^{t_K-1}$, $K/\boldsymbol{Q}$= a quadratic extension, $h^+_K$= the class number of $K$ in the narrow sense, $h^*_K$= the number of classes in a genus, $t_K$= the number of rational primes ramified in $K/\boldsymbol{Q}$) on the genera of quadratic forms may be described as establishing an equality between $E(K/\boldsymbol{Q})$ and other arithmetical invariants of $K$. Our general arithmetic theory of tori, i.e. the theory of isogenies, class number formulas, Tamagawa numbers, etc. furnishes us with general tools to determine $E(K/k)$ for any Galois extension ([1], [2], [4]).

In our Theorem below the following notation is used.

$K'/k$: the maximal abelian subextension of a Galois extension $K/k$,
$\mathfrak{g}$: the Galois group of $K/k$,
$\mathfrak{g}_V$: the Galois group of $K_V/k_v$ for $V|v$,
$K^\times$: the multiplicative group of $K$,
$\mathfrak{O}_V$: the ring of $\mathfrak{P}$-adic integers of $K_V$ when $V=\mathfrak{P}$, the field $K_V$ when $V$ is archimedean,
$\mathfrak{O}_V^\times$: the group of units of $\mathfrak{O}_V$,
$K_A^\times$: the idele group of $K$,
$\mathfrak{O}_K^\times$: the group of units of the ring $\mathfrak{O}_K$ of integers of $K$,
$H^0(G, A)=A^G/NA$: the 0-th Tate cohomology group for a finite group $G$ and a $G$-module $A$,
$[*]$: the cardinality of a set $*$.

**Theorem.**

$$(\sharp)\qquad E(K/k)=\frac{[\mathrm{Ker}\,(H^0(\mathfrak{g}, K^\times)\rightarrow H^0(\mathfrak{g}, K_A^\times))]\prod_v [H^0(\mathfrak{g}_v, \mathfrak{O}_V^\times)]}{[K':k][H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}.$$

**Corollary.** When $K/k$ is cyclic, we have

$$E(K/k)=\frac{\prod_v e_v(K/k)}{[K:k][H^0(\mathfrak{g}, \mathfrak{O}_K^\times)]}$$

where $e_v(K/k)$ means the ramification index for $K_V/k_v$, $V|v$.

**Remark.** In [3] I have treated the case where $K/k$ is a cyclic Kummer extension. Professor F. Sato informed me of miscalculations in [3]. Accordingly, formulas (6)–(12) in [3] need appropriate changes in view of more general formula ($\sharp$).

The proofs of propositions in this report will be published elsewhere.

[1] T. Ono: Arithmetic of algebraic tori. Ann. of Math., **74**, 101–139 (1961).
[2] ——: On the Tamagawa number of algebraic tori. ibid., **78**, 47–73 (1963).
[3] ——: A generalization of Gauss' theorem on the genera of quadratic forms. Proc. Japan Acad., **61A**, 109–111 (1985).
[4] Jih-Min Shyr: Class number formulas of algebraic tori with applications to relative class numbers of certain relative quadratic extensions of algebraic number fields. Thesis, The Johns Hopkins Univ., Baltimore, Maryland (1974).