83. Some Results on Bessel Processes

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Let α be any real number. The Bessel process with index α is the diffusion process on the half line $\boldsymbol{R}^+=[0, \infty)$, whose infinitesimal generator agrees with the differential operator

$$A=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\alpha-1}{x}\cdot\frac{d}{dx}\right). \tag{1}$$

We note that the formula (1) for the generator implies that the boundary point 0 is

entrance not exit for $\alpha\geqq 2$
entrance and exit for $0<\alpha<2$
exit not entrance for $\alpha\leqq 0$.

Thus for $\alpha\geqq 2$ or $\alpha\leqq 0$, the processes are completely specified by the generator (1) above, but for $0<\alpha<2$ appropriate boundary condition must be imposed at the origin. In this note we deal with two types of the boundaries, i.e., reflecting barrier and absorbing barrier.

Following Yosida [4], we define the generalized potential operator V for the semigroup T_t by

$$Vf=\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}T_t f\,dt \qquad (\lambda>0). \tag{2}$$

The representation of the potential operators associated with Bessel processes which is shown in [1], [2], will be stated here as

Proposition 1 (Reflecting case: Theorem 3 of Arakawa-Takeuchi [1]). *Assume that* $xf(x)\in L^1(\boldsymbol{R}^+)$ *for* $\alpha>0$, $\alpha\neq 2$ *and* $xf(x)\log x\in L^1(\boldsymbol{R}^+)$ *for* $\alpha=2$.

(i) *For* $0<\alpha\leqq 2$, *a necessary and sufficient condition for* $f\in\mathcal{D}(V)$ *is*

$$\int_0^\infty x^{\alpha-1}f(x)dx=0. \tag{3}$$

If $f\in\mathcal{D}(V)$, *then we have*

$$Vf(x)=2\int_0^\infty U(x\vee y)y^{\alpha-1}f(y)dy \tag{4}$$

with the kernel

$$U(x)=\begin{cases}\dfrac{1}{\alpha-2}\cdot\dfrac{1}{x^{\alpha-2}} & \text{if } 0<\alpha<2 \text{ and } \alpha>2\\[2ex] \log\dfrac{1}{x} & \text{if } \alpha=2,\end{cases} \tag{5}$$

here $x\vee y$ *denotes the greater of* x *and* y.

(ii) *For* $\alpha>2$, *a function* f *such that* $xf(x)\in L^1(\boldsymbol{R}^+)$ *is contained in* $\mathcal{D}(V)$, *and* $Vf(x)$ *is expressed by* (4) *with* (5).

Proposition 2 (Absorbing case: Takeuchi [2]). *For* $-\infty<\alpha<2$, *a* *function f such that $xf(x) \in L^1(\boldsymbol{R}^+)$ is contained in $\mathscr{D}(V)$, and the potential is given by*

$$Vf(x) = -2\int_0^\infty U(x \wedge y) y^{\alpha-1} f(y) dy, \tag{6}$$

here $U(x)$ has the representation

$$U(x) = \frac{1}{\alpha-2} \cdot \frac{1}{x^{\alpha-2}}$$

and $x \wedge y$ denotes the smaller of x and y.

Our main interest here is to show how the representations of (4) and (6) of the potential operators will be useful to know the probabilistic properties of the Bessel processes.

Let $p(t, x, y)$ be the transition probability density of the process and put

$$G(x, y) = \int_0^\infty p(t, x, y) dt. \tag{7}$$

Recall the definition (2) of Yosida potential. Noting that $e^{-\lambda t}$ is increasing as $\lambda \downarrow 0$, we have

$$\lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} T_t f(x) dt = \int_0^\infty T_t f(x) dt = \int_0^\infty f(y) G(x, y) dy. \tag{8}$$

The above equality (8) is always true regardless of whether the middle term is finite or not.

Lemma. *Assume the point 0 is reflecting barrier and $\alpha > 2$, then we have*

$$\int_0^r G(x, y) dy = \frac{r^2}{\alpha-2} - \frac{x^2}{\alpha}, \qquad x \leqq y.$$

Assume the point 0 is absorbing barrier and $\alpha < 0$, then

$$\int_r^\infty G(x, y) dy = \frac{r^2}{\alpha-2} - \frac{x^2}{\alpha}, \qquad r \leqq x.$$

Proof. Note that in the reflecting case

$$Vf(x) = 2\int_0^\infty U(x \vee y) y^{\alpha-1} f(y) dy. \tag{9}$$

Put

$$f(y) = \begin{cases} 1 & 0 \leqq y \leqq r \\ 0 & r \leqq y \end{cases}$$

in (8) and (9), then we get

$$\int_0^r G(x, y) dy = 2\int_0^r U(x \vee y) y^{\alpha-1} dy = \frac{2}{\alpha-2} \int_0^r \frac{y^{\alpha-1}}{(x \vee y)^{\alpha-2}} dy$$

from which the first formula follows by simple calculations.

By the same argument as above, the formula for the absorbing case is obtained from

$$\int_r^\infty G(x, y) dy = -\frac{2}{\alpha-2} \int_r^\infty \frac{y^{\alpha-1}}{(x \wedge y)^{\alpha-2}} dy.$$

The integral

$$\int_0^r G(x, y) dy \quad \text{and} \quad \int_r^\infty G(x, y) dy$$

computed above, is the expected time spent in the interval $[0, r]$ and the interval $[r, \infty) \subset \boldsymbol{R}^+$ by the Bessel process $X(t)$ starting at x, respectively.

Let T_r denote the first hitting time of the point r by the Bessel process $X(t)$, that is,

$$T_r = \inf\{t > 0 : X(t) = r\}.$$

Theorem 1. (i) *Reflecting case*: *Suppose $\alpha > 0$ and $x \leqq r$.*

(ii) *Absorbing case*: *Suppose $\alpha < 0$ and $r \leqq x$.*

Then, in either case, the expected time for reaching at the point r is as follows,

$$E_x(T_r) = \frac{r^2 - x^2}{\alpha}.$$

Proof. A proof for the reflecting case is given in [3]. Unfortunately the technique in [3] is not applicable to the absorbing case. The method here has a fine unity for both cases. On the other hand, the approach by potential kernel does not apply to the case $0 < \alpha < 2$ due to restriction (3).

The usual first passage arguments yield the fundamental identity

$$G(x, y) = G_B(x, y) + \int H_B(x, dz) G(z, y), \qquad x, y \in \boldsymbol{R}^+, \tag{10}$$

where $G_B(x, y)$ is the density of the measure

$$\int_0^\infty P_x(T_B > t, X(t) \in dy) dt$$

and $H_B(x, d\xi)$ is the hitting measure of the set B

$$H_B(x, d\xi) = P_x(X(T_B) \in d\xi, T_B < \infty). \tag{11}$$

Put $B = \{r\}$ in the fundamental identity (10), then we get

$$E_x(T_r) = \int_0^r G_r(x, y) dy = \int_0^r G(x, y) dy - \int_0^r G(r, y) dy. \tag{12}$$

The desired result now follows easily from Lemma.

As for the absorbing barrier case, the identity (12) can be rewritten as

$$E_x(T_r) = \int_r^\infty G_r(x, y) dy = \int_r^\infty G(x, y) dy - \int_r^\infty G(r, y) dy.$$

Again from Lemma, we obtain the conclusion.

Now we introduce the potential kernel $U(x, y)$ of the Bessel processes by

$$Uf(x) = \int_0^\infty U(x, y) y^{\alpha - 1} f(y) dy.$$

Theorem 2. *Let σ_r be the delta probability measure at the point r.*

(i) *For the reflecting case, let $\alpha > 2$. Then $I_r = [0, r]$ has equilibrium measure $\mu_r = ((\alpha - 2)/2) r^{\alpha - 2} \cdot \sigma_r$, capacity $C(I_r) = ((\alpha - 2)/2) r^{\alpha - 2}$ and equilibrium potential given by*

$$U\mu_r(x) = P_x(T_r < \infty) = \left(\frac{r}{x}\right)^{\alpha - 2} \wedge 1.$$

(ii) *For the absorbing case, let $-\infty < \alpha < 2$. Then $J_r = [r, \infty)$ has equilibrium measure $\mu_r = ((2 - \alpha)/2) r^{\alpha - 2} \cdot \sigma_r$, capacity $C(J_r) = ((2 - \alpha)/2) r^{\alpha - 2}$ and equilibrium potential given by*

$$U\mu_r(x)=P_x(T_r<\infty)=\left(\frac{x}{r}\right)^{2-\alpha}\wedge 1.$$

Proof. We first prove $P_x(T_r<\infty)=(r/x)^{\alpha-2}\wedge 1$ for reflecting case. If $x\leqq r$, the result follows from Corollary to Proposition 1 in [3] and continuity of path. Suppose $r\leqq x$. Then, combining Proposition 1 in [3] with continuity of path, we see

$$P_x(T_r<\infty)=\lim_{y\to\infty}\frac{x^{2-\alpha}-y^{2-\alpha}}{r^{2-\alpha}-y^{2-\alpha}}=\left(\frac{r}{x}\right)^{\alpha-2}.$$

For the absorbing case, let $x<r$. Then in virtue of Proposition 1 in [3] and $P_x(T_0<\infty)=1$, we find

$$P_x(T_r<\infty)=P_x(T_0<\infty,\ T_r<T_0)=P_x(T_r<T_0)=\left(\frac{x}{r}\right)^{2-\alpha}.$$

For the case $x\geqq r$, the conclusion follows since $P_x(T_0<\infty)=1$.

To give an alternative proof, it can be shown for $\alpha>2$ in reflecting case that

$$P_x(T_r<\infty)=\frac{U(x,r)}{U(r,r)} \tag{13}$$

from which the conclusion follows. Also in absorbing case the above identity (13) is still useful.

Observe that

$$\int U(r,r)\sigma_r=\frac{2}{\alpha-2}r^{2-\alpha}. \tag{14}$$

We write the constant value (14) as c_r and use the notation H_r as analogue of (11). Then

$$\begin{aligned}\int U(x,y)\sigma_r(dy)&=\int H_r(x,y)\int U(r,r)\sigma_r(dy)\\ &=c_rP_x(T_r<\infty)=c_r\left(\frac{r}{x}\wedge 1\right)^{\alpha-2}.\end{aligned} \tag{15}$$

Let $\mu_r=((\alpha-2)/2)r^{\alpha-2}\sigma_r$. It follows, referring to (14) and (15), that

$$U\mu_r=P_\cdot(T_r<\infty)=1 \qquad \text{on } [0,r].$$

Therefore μ_r is the equilibrium measure of I_r. Other formulas can be obtained immediately.

References

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