# 76. *Locally Trivial Displacements of Analytic Subvarieties with Ordinary Singularities*

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§0. **Introduction.** In [2] K. Kodaira studied on *locally trivial* displacements of a surface $S$ with ordinary singularities in a threefold $W$, and proved that if $S$ is "*semi-regular*" in $W$, then there exists an effectively parametrized maximal family of *locally trivial* displacements of $S$ in $W$ whose parameter space is non-singular. In [7] M. Namba proved the existence of the universal family of *locally trivial* displacements of surfaces with ordinary singularities. It is a problem to extend these results to higher dimensional cases. The purpose of this note is to outline our recent results on this problem. Details will be published elsewhere.

§1. **Analytic families of locally trivial displacements of analytic subvarieties with ordinary singularities.** "*Ordinary*" singularities are those of the image $\pi(X)\subset \boldsymbol{P}^m(\boldsymbol{C})$ of a non-singular algebraic manifold $X$ embedded in a sufficiently higher dimensional projective space $\boldsymbol{P}^N(\boldsymbol{C})$ by a "*generic*" linear projection $\pi: \boldsymbol{P}^N(\boldsymbol{C})\to \boldsymbol{P}^m(\boldsymbol{C})$. In [4] J. N. Mather showed that if the pair $(n, m)$ ($n=\dim X$) of positive integers belongs to so-called "*nice range of dimensions*" (in the case of $m=n+1$, the pair $(n, m)$ is in the "*nice range of dimensions*" if and only if $n\leqq 14$), then the restriction $\pi|_X: X\to \boldsymbol{P}^m(\boldsymbol{C})$ to $X$ of a "*generic*" linear projection $\pi: \boldsymbol{P}^N(\boldsymbol{C})\to \boldsymbol{P}^m(\boldsymbol{C})$ is a *locally* (*infinitesimally*) *stable holomorphic map* (for definition see [3]). Making use of J. N. Mather's results concerning *stable map-germs*, we can give all possible defining equations of ordinary singularities in the case of some dimensions. However, our arguments do not depend on explicit descriptions of ordinary singularities by local coordinates. So we adopt the following as the definition of an "*analytic subvariety with ordinary singularities*".

Definition 1. Let $Z$ be a proper analytic subvariety of a pure dimension of a complex manifold $Y$, $\nu: X\to Z$ the normalization of $Z$, and let $f=\iota\circ\nu: X\to Y$ be the composition of the normalization $\nu: X\to Z$ and the inclusion map $\iota: Z\hookrightarrow Y$. We say $Z$ is an *analytic subvariety with ordinary singularities of* $Y$ if the following are satisfied:

(i) $X$ is non-singular,

(ii) $f=\iota\circ\nu: X\to Y$ is a locally (infinitesimally) stable holomorphic map.

From now on, let $Z$ be an irreducible analytic subvariety with ordinary singularities in a compact complex manifold $Y$, and let $f: X\to Y$ be the composition of the normalization $\nu : X \rightarrow Z$ and the inclusion map $\iota : Z \hookrightarrow Y$. We fix these notations. An "*analytic family* $(Y \times M, \mathscr{Z}, \pi, M, o)$ *of locally trivial displacements of* $Z$ *in* $Y$ *parametrized by an analytic variety* $M$" is defined as in the case of surfaces (cf. [2]).

We denote by $\Theta_Y$ the sheaf of germs of holomorphic tangent vector fields on $Y$, and by $\Theta_Y(\log Z)$ the sheaf of germs of *logarithmic tangent vector fields along* $Z$ on $Y$, i.e., the subsheaf of $\Theta_Y$ consisting of derivations of $\mathcal{O}_Y$ which send $\mathscr{I}(Z)$, the ideal sheaf of $Z$ in $\mathcal{O}_Y$, into itself.

**Definition 2.** We denote $\mathcal{N}_{Z/Y}$ the quotient sheaf $\Theta_Y / \Theta_Y(\log Z)$, and call this the *sheaf of infinitesimal locally trivial displacements of* $Z$ *in* $Y$.

**Proposition 1.** *For a family* $(Y \times M, \mathscr{Z}, \pi, M, o)$ *of locally trivial displacements of* $Z$ *in* $Y$ *parametrized by an analytic variety* $M$, *we can define a so-called characteristic map*

$$\sigma_0 : T_0(M) \longrightarrow H^0(Z, \mathcal{N}_{Z/Y})$$

($T_0(M)$ *denotes the Zariski tangent space of* $M$ *at* $o$), *which is the same one defined by K. Kodaira in* [2] *if* $Z$ *is a surface.*

**§ 2. Main theorem and concluding results.** We denote by $\mathscr{D}(f)$ resp. $\mathscr{L}(Z)$, the category of isomorphism classes of germs of families of deformations of the holomorphic map $f: X \rightarrow Y$ resp. the category of isomorphism classes of germs of families of locally trivial displacements of $Z$ in $Y$. For a germ of a family $D = (\mathscr{X}, F, \pi, M, o)$ of deformations of the holomorphic map $f: X \rightarrow Y$ parametrized by an analytic variety $M$ (for definition see [1]) resp. a germ of a family $L = (Y \times M, \mathscr{Z}, \pi, M, o)$ of locally trivial displacements of $Z$ in $Y$, we denote by $\{D\} \in Ob(\mathscr{D}(f))$ (the objects of $\mathscr{D}(f)$) resp. $\{L\} \in Ob(\mathscr{L}(Z))$ the isomorphism class of $D$ resp. of $L$.

**Main theorem.** *For a germ of a family* $D = (\mathscr{X}, F, \pi, M, o)$ *of deformations of the map* $f: X \rightarrow Y$, *we denote by* $\mathscr{F}(D)$ *the germ of the family* $(Y \times M, F(\mathscr{X}), \varpi, M, o)$ *of analytic subvarieties of* $Y$, *where* $\varpi$ *denotes the restriction to* $F(\mathscr{X})$ *of the canonical projection* $Pr_M : Y \times M \rightarrow M$. *Then*:

(i) $\mathscr{F}(D)$ *is a germ of a family of locally trivial displacements of* $Z$ *in* $Y$.

(ii) *For two germs of families* $D$ *and* $D'$ *of deformations of the map* $f: X \rightarrow Y$, $D$ *is isomorphic to* $D'$ (*as germs*) *if and only if* $\mathscr{F}(D)$ *is isomorphic to* $\mathscr{F}(D')$.

(iii) *The correspondence* $\{D\} \in Ob(\mathscr{D}(f)) \rightarrow \{\mathscr{F}(D)\} \in Ob(\mathscr{L}(Z))$ *gives rise to an isomorphism between the categories* $\mathscr{D}(f)$ *and* $\mathscr{L}(Z)$.

(iv) *There exists an isomorphism* $\ell : H^0(Z, \mathcal{N}_{Z/Y}) \rightarrow H^0(X, \mathcal{T}_{X/Y})$ *of cohomology groups such that, for any family* $D = (\mathscr{X}, F, \pi, M, o)$ *of deformations of the map* $f: X \rightarrow Y$, *the following diagram commutes*:

$$\begin{array}{ccc} & \overset{\tau_0}{\nearrow} & H^0(X, \mathcal{T}_{X/Y}) \\ T_0(M) & & \uparrow -\ell \\ & \underset{\sigma_0}{\searrow} & H^0(Z, \mathcal{N}_{Z/Y}) \end{array} ,$$

*where* $\mathcal{T}_{X/Y}$ *denotes the sheaf of germs of infinitesimal deformations of the map* $f: X \to Y$, $\tau_0: T_0(M) \to H^0(X, \mathcal{T}_{X/Y})$ *the characteristic map of the family* $D$ *(for definition see* [1]*) and* $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ *that of the family of* $\mathcal{F}(D) = (Y \times M, F(\mathcal{X}), \varpi, M, o)$.

Combining above Main theorem and the results in [5], [6], we have the following propositions:

**Proposition 2.** *For an irreducible analytic subvariety* $Z$ *with ordinary singularities of a compact complex manifold* $Y$, *there exists a family* $(Y \times M, \mathcal{Z}, \pi, M, o)$ *of locally trivial displacements of* $Z$ *in* $Y$ *such that:*

(i) *The characteristic map* $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ *is injective,*

(ii) *the family is maximal at any point* $t \in M$,

(iii) *the family is universal at* $o$.

*Furthermore, if* $H^1(Z, \mathcal{N}_{Z/Y}) = 0$, *then* $M$ *is non-singular and* $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ *is bijective.*

**Proposition 3.** *Let* $Z$ *be an irreducible hypersurface with ordinary singularities in a compact complex manifold* $Y$. *Then there exists a family* $(Y \times M, \mathcal{Z}, \pi, M, o)$ *of locally trivial displacements of* $Z$ *in* $Y$ *such that:*

(i) *The characteristic map* $\sigma_t: T_t(M) \to H^0(Z_t, \mathcal{N}_{Z_t/Y})$ *is injective for any point* $t \in M$,

(ii) *the family is universal at any point* $t \in M$.

*Furthermore, if* $H^1(Z, \mathcal{N}_{Z/Y}) = 0$, *then* $M$ *is nonsingular and* $\sigma_t: T_t(M) \to H^0(Z_t, \mathcal{N}_{Z_t/Y})$ *is bijective for* $t \in M$ *sufficiently close to* $o$.

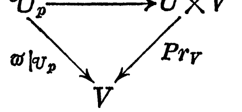
**§3. The proof of Main theorem.** We deduce Main theorem by the following two theorems.

**Theorem 1.** *Let* $(\mathcal{X}, F, \pi, M, o)$ *be a family of deformations of a locally (infinitesimally) stable holomorphic map* $f: X \to Y$ *with* $Y$ *compact* *(*$X$ *is not necessary to be compact). We define the set*

$$\Sigma := \{p \in \mathcal{X} \mid (dF)_p : T_p(\mathcal{X}) \longrightarrow T_{F(p)}(Y \times M) \text{ is not surjective}\},$$

*which is equipped with the structure of a reduced complex space. We assume that* $F|_\Sigma : \Sigma \to Y \times M$ *is a proper map. Then there exists an open neighborhood* $M'$ *of* $o$ *in* $M$ *such that for any* $t \in M'$ *the map* $F_t: X_t \to Y$ *is a locally (infinitesimally) stable holomorphic map.* (Note that $\Sigma = \mathcal{X}$ in case that $\dim X < \dim Y$.)

**Theorem 2** (Relative normalization theorem). *Let* $\varpi: \mathcal{Z} \to M$ *be an analytic family of analytic varieties parametrized by an analytic variety* $M$, *which is locally trivial at every point in* $\mathcal{Z}$ *in the following sense; for each point* $p \in \mathcal{Z}$, *there exists an open neighborhood* $\mathcal{U}_p \subset \mathcal{Z}$ *of* $p$ *and an isomorphism* $\phi_p: \mathcal{U}_p \to U \times V$, *where* $U = \mathcal{U}_p \cap Z_{\varpi(p)}$ $(Z_{\varpi(p)} := \varpi^{-1}(\varpi(p))$ *and* $V = \varpi(\mathcal{U}_p)$, *such that the following diagram*



*is commutative.*

*Then there exists an analytic family* $\pi : \mathcal{X} \to M$ *of complex manifolds parametrized by the analytic variety* $M$, *and a surjective holomorphic map* $\nu : \mathcal{X} \to \mathcal{Z}$ *over* $M$ (i.e. $\pi = \varpi \circ \nu$) *satisfying the following conditions:*

($\alpha$) $\nu_t : X_t \to Z_t$ *is the normalization of* $Z_t$ *for any* $t \in M$ ($X_t := \pi^{-1}(t)$, $\nu_t := \nu|_{X_t} : X_t \to Z_t$),

($\beta$) *the map* $\nu : \mathcal{X} \to \mathcal{Z}$ *is locally trivial in the following sense;*
*for any point* $p \in \mathcal{Z}$, *there exist an open neighborhood* $\mathcal{U}_p$ *of* $p$ *in* $\mathcal{Z}$, *an isomorphism* $\phi_p : \mathcal{U}_p \to U \times V$ *over* $V$, *where* $U = \mathcal{U}_p \cap Z_{\varpi(p)}$ *and* $V = \varpi(\mathcal{U}_p)$, *and an isomorphism* $\psi_p : \nu^{-1}(\mathcal{U}_p) \xrightarrow{\sim} U^* \times V$ *over* $V$, *where* $U^* = \nu^{-1}(\mathcal{U}_p) \cap X_{\varpi(p)}$, *such that the diagram*

$$\begin{array}{ccc} \nu^{-1}(\mathcal{U}_p) & \xrightarrow{\psi_p} & U^* \times V \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu_{\varpi(p)} \times id_V} \\ \mathcal{U}_p & \xrightarrow{\phi_p} & U \times V \end{array}$$

*is commutative.*

*Furthermore, the above family* $\pi : \mathcal{X} \to M$ *and the surjective holomorphic map* $\nu : \mathcal{X} \to \mathcal{Z}$ *over* $M$ *are uniquely determined up to isomorphisms over* $M$.

## References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