74. *On an Euler Product Ring*

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§1. **Euler product rings.** Let Z be the ring of rational integers. We denote by $E(Z)$ the (universal) completion $\hat{Z}$ of Z. Hence, denoting the ring of p-adic integers by Z_p, we have a canonical isomorphism $E(Z)\cong\prod_p Z_p$, where p runs over all rational primes. We consider $E(Z)$ as an "Euler product ring" (over Z) via this infinite product expression; see Theorem 1 below for another explanation. In this paper we note some properties of $E(Z)$ related to the structure of maximal ideals of $E(Z)$ in a bit generalized situation. A detailed study will appear elsewhere.

We fix the notation. Let A be a commutative ring with 1. We define: $E(A)=A\otimes_Z E(Z)$. We denote by $\mathrm{Max}(A)$ the space of all maximal ideals of A, which is equipped with the Stone topology. For $q\in\mathrm{Max}(Z)\cup\{0\}$ we put

$$\mathrm{Max}_q(A)=\{M\in\mathrm{Max}(A);\ \text{the characteristic of } A/M \text{ is } q\}.$$

We say that $M\in\mathrm{Max}(A)$ is *cofinite* if A/M is a finite field, and define the norm $N(M)$ of M via $N(M)=\sharp(A/M)$, where $\sharp$ denotes the cardinality. We denote by $\mathrm{Max}^{cf}(A)$ the set consisting of all cofinite maximal ideals of A. Obviously we have:

$$\mathrm{Max}^{cf}(A)\subset\mathrm{Max}(A)-\mathrm{Max}_0(A)=\mathrm{Max}_2(A)\cup\mathrm{Max}_3(A)\cup\cdots.$$

We define the zeta function $\zeta(s, A)$ of A (at least formally) by the following Euler product $\zeta(s,A)=\prod_M(1-N(M)^{-s})^{-1}$ where M runs over $\mathrm{Max}^{cf}(A)$ and s is a complex number; this zeta function coincides with the zeta function $\zeta(s, M(A))$ of the category $M(A)$ of A-modules in the sense of [5]. (We note that some details of [5] are appearing in Proc. London Math. Soc.) We denote by $\Omega(A)$ the A-module of absolute Kähler differentials of A (over Z); we refer to Grothendieck [2; Chap. 0, §20] concerning Kähler differentials.

Hereafter, let $A=O_F$ be the integer ring of a finite number field F. Then $E(A)\cong\hat{A}\cong\prod_{\mathfrak{p}}A_{\mathfrak{p}}$, where $\hat{A}$ and $A_{\mathfrak{p}}$ denote respectively the completion and $\mathfrak{p}$-adic completion of A, and $\mathfrak{p}$ runs over $\mathrm{Max}(A)$. We have:

Theorem 1. $\zeta(s, E(A))=\zeta(s, A)$.

Theorem 2. *$\mathrm{Max}(E(A))$ is a compact Hausdorff space.*

Theorem 3. $\Omega(E(A))\neq 0$.

Remark 1. (1) $\zeta(s, A)$ is equal to the Dedekind zeta function of F. (2) $\mathrm{Max}(A)$ is not a Hausdorff space. (3) $\Omega(A)=0$.

§2. **Proofs.** First we show

Theorem 1a. $\mathrm{Max}_p(E(A))=\{\mathfrak{p}E(A);\mathfrak{p}\in\mathrm{Max}(A),\mathfrak{p}|p\}$ *for each rational* prime p.

Proof. Let $M \in \mathrm{Max}_p(E(A))$. Then $p \in M$, since $E(A)/M$ is of characteristic p. Put $\boldsymbol{p}=M \cap A$. Then $\boldsymbol{p}$ is a prime ideal of A (since M is a prime ideal of $E(A)$) containing p. Hence $\boldsymbol{p} \in \mathrm{Max}(A)$ and $\boldsymbol{p}|p$. Moreover $\boldsymbol{p}E(A) \subset M \subset E(A)$ and $E(A)/\boldsymbol{p}E(A) \cong A/\boldsymbol{p}$ since $\boldsymbol{p}E(A) \cong \boldsymbol{p}A_{\boldsymbol{p}} \times \prod_{\boldsymbol{l} \neq \boldsymbol{p}} A_{\boldsymbol{l}}$, where $\boldsymbol{l}$ runs over $\mathrm{Max}(A)-\{\boldsymbol{p}\}$. In particular, both $\boldsymbol{p}E(A)$ and M are maximal ideals of $E(A)$. Hence $M=\boldsymbol{p}E(A)$. Q.E.D.

Proof of Theorem 1. From the proof of Theorem 1a we see that

$$\mathrm{Max}^{cf}(E(A))=\bigcup_p \mathrm{Max}_p(E(A))=\{\boldsymbol{p}E(A)\,;\,\boldsymbol{p} \in \mathrm{Max}(A)\}$$

and $N(\boldsymbol{p}E(A))=N(\boldsymbol{p})$ for each $\boldsymbol{p} \in \mathrm{Max}(A)$. Hence we have $\zeta(s, E(A)) =\zeta(s, A)$. Q.E.D.

Hereafter we denote by $^*A$ a good nonstandard model of A as in Robinson [6], where a surjective ring homomorphism $^*A \to E(A)$ is constructed. We use a fact that $\mathrm{Max}(^*A)$ is a compact Hausdorff space, which follows from Cherlin [1] (cf. Klingen [3]) where $\mathrm{Max}(^*A)$ is parametrized via certain ultra-filters.

Theorem 2a. *Let E be a commutative ring with* 1 *having a surjective ring homomorphism $^*A \to E$. Then* $\mathrm{Max}(E)$ *is a compact Hausdorff space.*

Proof. It is easy to see that $\mathrm{Max}(E)$ is (considered to be) a subspace of $\mathrm{Max}(^*A)$. Q.E.D.

Proof of Theorem 2. Apply Theorem 2a to Robinson's surjective ring homomorphism $^*A \to E(A)$. Q.E.D.

We put $E_0(A)=\prod_{\boldsymbol{p}}(A/\boldsymbol{p})$ where $\boldsymbol{p}$ runs over $\mathrm{Max}(A)$.

Theorem 3a. *Let E be a commutative ring with* 1 *having a surjective ring homomorphism $E \to E_0(A)$. Then $\Omega(E) \neq 0$.*

Proof. Since there is a surjective $E_0(A)$-module homomorphism ([2; Chap. 0, 20. 5. 12]) $\Omega(E) \otimes_E E_0(A) \to \Omega(E_0(A))$, it is sufficient to show that $\Omega(E_0(A)) \neq 0$. Take an $M \in \mathrm{Max}_0(E_0(A))$. Then we see that $E_0(A)/M$ is a transcendental extension field of the rational number field $\boldsymbol{Q}$ since $\sharp(E_0(A)/M)=\aleph$ by Kochen [4, Th. 6. 5 and Th. 8. 1]. Hence $\Omega(E_0(A)/M) \neq 0$ ([2; Chap. 0, 20. 6. 20]). Thus, using the surjective homomorphism

$$\Omega(E_0(A)) \underset{E_0(A)}{\otimes} (E_0(A)/M) \longrightarrow \Omega(E_0(A)/M)$$

we see that $\Omega(E_0(A)) \neq 0$. Q.E.D.

Proof of Theorem 3. Since there is a canonical surjective ring homomorphism $E(A) \to E_0(A)$, Theorem 3 follows from Theorem 3a. Q.E.D.

Remark 2. From the above proofs, it is easy to see that if $E=\prod_{\boldsymbol{p}} E_{\boldsymbol{p}}$ with $E_{\boldsymbol{p}}=A_{\boldsymbol{p}}$ or $A/\boldsymbol{p}$, where $\boldsymbol{p}$ runs over $\mathrm{Max}(A)$ for $A=O_F$, then Theorems 1–3 hold for E (for example: $E=E_0(A)$) instead of $E(A)$. Moreover $\mathrm{Max}(E_0(A))$ is homeomorphic to the Stone-Čech compactification of $\mathrm{Max}(A)_d$, the discrete version of $\mathrm{Max}(A)$ (cf. Kochen [4, Th. 8.1]). We remark also that $\Omega(^*A) \neq 0$ by Theorem 3a.

§ 3. Modifications. Let $A=O_F$ be as above. For a commutative ring R with 1 we put $E_R(A)=E(A) \otimes_Z R=A \otimes_Z E_R(Z)$. We have analogous results for $E_R(A)$ also. For simplicity, here we note

Theorem 3b. *$\Omega(E_R(A))\neq 0$ if $R\supset \boldsymbol{Q}$.*

Proof. Since there is an *injective* homomorphism ([2; Chap. 0, 20.5.5])

$$\Omega(E_{\boldsymbol{Q}}(A))\otimes_{\boldsymbol{Q}} R \longrightarrow \Omega(E_R(A)),$$

it is sufficient to show that $\Omega(E_{\boldsymbol{Q}}(A))\neq 0$. Take an $l\in \mathrm{Max}\,(A)$ and let $M(l)$ be the maximal ideal of $E_{\boldsymbol{Q}}(A)$ consisting of elements with zero l-components. Then $E_{\boldsymbol{Q}}(A)/M(l)\cong Q(A_l)$, the quotient field of A_l, so $\Omega(E_{\boldsymbol{Q}}(A)/M(l))\neq 0$. Hence $\Omega(E_{\boldsymbol{Q}}(A))\neq 0$ as before. Q.E.D.

Remark 3. From this proof we see that the module $\Omega_R(E_R(A))$ of relative Kähler differentials over R is non-zero. We note that $E_{\boldsymbol{C}}(A)$ is particularly interesting in connection with the following: (1) the complex valued functions on $\mathrm{Max}\,(E_{\boldsymbol{C}}(A))$ and (2) the natural homomorphism $\mathrm{Aut}\,(E_{\boldsymbol{C}}(A))\rightarrow \mathrm{Aut}\,(\mathrm{Max}\,(E_{\boldsymbol{C}}(A)))$.

The following is another modification.

Theorem 1c. *Let A be a subring of $\boldsymbol{Q}$. Then $\zeta(s, E(A))=\zeta(s, A)$.*

Proof. There is a subset S of $\mathrm{Max}(\boldsymbol{Z})$ such that $A=\boldsymbol{Z}[S^{-1}]$, where $S^{-1}=\{p^{-1}; p\in S\}$. Then, as in the proof of Theorem 1, we see that $\zeta(s, E(A)) =\prod_{p\notin S}(1-p^{-s})^{-1}=\zeta(s, A)$. (Remark that if $A=\boldsymbol{Z}$ and $\boldsymbol{Q}$ then $S=\phi$ and $\mathrm{Max}\,(\boldsymbol{Z})$ respectively, and $\zeta(s, \boldsymbol{Q})=1$ by our definition.) Q.E.D.

The analytic behaviour of this zeta function (which is equal to $\zeta(s, \boldsymbol{Z})\prod_{p\in S}(1-p^{-s})$) does not seem to be so clear when both S and $\mathrm{Max}(\boldsymbol{Z})-S$ are infinite sets. We obtain the following result by a modification of the method of [5].

Theorem 4. *Let χ be a Dirichlet character of $\boldsymbol{Z}$ of order 2. Put $S=\{p\in \mathrm{Max}(\boldsymbol{Z}); \chi(p)\neq 1\}$ and $A=\boldsymbol{Z}[S^{-1}]$. Then $\zeta(s, A)$ is continued to be an analytic function with singularities in $\mathrm{Re}(s)>0$ with the natural boundary $\mathrm{Re}(s)=0$.*

More generally:

Theorem 4a. *Let χ be a Dirichlet character of $\boldsymbol{Z}$ of order 2. Let $X=\mathrm{Max}\,(\boldsymbol{Z}[T_1, \cdots, T_r])$ for $r\geqq 0$ where $T_1, \cdots, T_r$ are indeterminates. (If $r=0$, $X=\mathrm{Max}\,(\boldsymbol{Z})$.) Put $X_+=\{x\in X; \chi(N(x))=1\}$ and $X_-=\{x\in X; \chi(N(x))=-1\}$. Then the zeta functions $\zeta(s, X_+)$ and $\zeta(s, X_-)$ are analytic (with singularities) in $\mathrm{Re}\,(s)>0$ with natural boundaries $\mathrm{Re}\,(s)=0$.*

A simple example of such a zeta function is $\prod_{p\equiv 1 \bmod 3}(1-p^{-s})^{-1}$, where 3 can be replaced by 4 and 6 also.

As another application of [5] we note that each Hardy-Littlewood constant can be "identified" with the leading coefficient of the Laurent expansion at $s=1$ of a naturally associated Euler product treated in [5-I, Theorem 1]; Hardy-Littlewood constants appeared in the famous Hardy-Littlewood conjectures published as "Partitio Numerorum III" in 1922, and these constants describe the distribution of prime values of polynomials (twin primes, primes of the form $n^2+1, \cdots$) and the generalized Goldbach problem.

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