71. *The Regularity of Discrete Models of the Boltzmann Equation*

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The aim of this note is to single out a category of the discrete Boltzmann equations as the regular models of the Boltzmann equation. It is shown that there exist regular models with n moduli of velocities for an arbitrary integer $n \geq 2$.

1. Let $M=\{v_1, \cdots, v_d\}$ be the set of velocities, i.e., the constant vectors in $\boldsymbol{R}^3$. We assume that the linear span of M coincides with $\boldsymbol{R}^3$. The model M is essentially three-dimensional in this sense. First of all, we introduce the notion of the collision. Let us denote by Σ the set of all unordered pairs of distinct velocities. We may set

$$\Sigma=\{(v_i, v_j)\,;\, 1 \leq i < j \leq d\}.$$

Let $\alpha, \beta \in \Sigma$. Then the ordered pair of α and β is called a collision, if

(i) $\alpha \neq \beta$,

(ii) the momentum of α equals the momentum of β,

(iii) the energy of α equals the energy of β.

It is usual to denote the ordered pair by $\alpha \rightarrow \beta$. We call α and β the initial and the final states of the collision, respectively. It is assumed in the following that there exists at least one collision. Now let $\mathcal{C}$ be the set of all collisions. We obtain a partition of $\mathcal{C}$ by the equivalence relation given below.

We introduce the group of transformations acting in M. We set

$$\tilde{G}=\{T\,;\, T \in \mathcal{O}(\boldsymbol{R}^3),\ TM=M\}.$$

Here, $\mathcal{O}(\boldsymbol{R}^3)$ denotes the group of orthogonal transformations. $\tilde{G}$ induces naturally a group of isometric transformations on M, which we denote by G. It is easily seen that G is determined uniquely as the maximal set of isometric transformations on M. We define that $\alpha \rightarrow \beta$ and $\alpha' \rightarrow \beta'$ are equivalent if these collisions are obtained from each other by performing a transformation which belongs to G or by interchanging the initial and the final states or by combining these two operations. The constants A_{ij}^{kl} appearing in the definition of the collision term may be identified with a "step function" subordinate to the partition of $\mathcal{C}$, which is induced from the equivalence relation given above. (See [3] for details.) Thus, if $\mathcal{C}$ consists of m equivalence classes, we have m arbitrary constants in defining the collision term. The general form of the discrete Boltzmann equation is given by

$$(1) \qquad \frac{\partial F_i}{\partial t}+v_i\cdot \nabla_x F_i=\frac{1}{2}\sum_{j,k,l}\{A^{ij}_{kl}F_kF_l-A^{kl}_{ij}F_iF_j\}, \qquad i=1,2,\cdots,d.$$

Here A^{kl}_{ij} is set to be zero if the formal expression $(v_i, v_j)\to(v_k, v_l)$ does not correspond to a collision.

We say that (1) is regular if the following properties hold:

1°) The equation (1) is irreducible in the sense that the system can not be decomposed into two decoupled subsystems.

2°) The collision term on the right side of (1) is invariant under the associated transformation group G.

3°) The stability condition for Maxwellians is satisfied. (We refer the reader to [3] for the precise statement of this condition.)

Note that 2°) is always satisfied when A^{kl}_{ij} is chosen according to the procedure described above. The other conditions 1°) and 3°) can be verified without the knowledge of A^{kl}_{ij}. Hence we may also say that the discrete model M is regular by abuse of the terminology.

2. Our results are summarized in the following two theorems. We denote by $\mathcal{R}(G)$ the set of regular models with the prescribed transformation group G. For the notations such as T, T_d, T_h, O_h, I_h, see, for example, [1].

Theorem 1. *Let* $n\geq 2$ *be an arbitrary integer and let* G *be either* T_d *or* O_h. *Then* $\mathcal{R}(G)$ *contains a model with* n *moduli of velocities.*

Theorem 2. *Let* $n\geq 2$ *be an arbitrary integer and let* G *be one of* T, T_h *and* I_h. *Then* $\mathcal{R}(G)$ *contains a model with* n *moduli of velocities.*

In order to prove Theorem 2, we need the computer. See [4] for details. On the contrary, Theorem 1 can be proved without using the computer.

Remark 1. For concrete models known up to date, the dimension of the space of summational invariants is five in the case of $G=T_d$, O_h, while the dimension of the same space turns out to be eight in the case of $G=T$, T_h, I_h. See [2], [3], [4]. It is not known whether these facts hold in general or not.

Remark 2. We obtain a 13-velocity model by omitting a vertex of the cube in the 14-velocity model studied in [3]. This model is also regular and we have $G=C_3$.

3. We give a sketch of the proof. Details are published elsewhere. First we consider the case where $G=O_h$. We define the 6, 12, 8, 6, 24-velocity models as follows. Let

$u^{(1)}_1=(1, 0, 0)$, $u^{(1)}_2=(0, 1, 0)$, $u^{(1)}_3=(0, 0, 1)$, $u^{(1)}_4=(-1, 0, 0)$,
$u^{(1)}_5=(0, -1, 0)$, $u^{(1)}_6=(0, 0, -1)$; $u^{(2)}_1=(1, 0, 1)$, $u^{(2)}_2=(0, 1, 1)$,
$u^{(2)}_3=(-1, 0, 1)$, $u^{(2)}_4=(0, -1, 1)$, $u^{(2)}_5=(-1, 0, -1)$, $u^{(2)}_6=(0, -1, -1)$,
$u^{(2)}_7=(1, 0, -1)$, $u^{(2)}_8=(0, 1, -1)$, $u^{(2)}_9=(1, 1, 0)$, $u^{(2)}_{10}=(-1, 1, 0)$,
$u^{(2)}_{11}=(-1, -1, 0)$, $u^{(2)}_{12}=(1, -1, 0)$; $u^{(3)}_1=(1, 1, 1)$, $u^{(3)}_2=(-1, 1, 1)$,
$u^{(3)}_3=(-1, -1, 1)$, $u^{(3)}_4=(1, -1, 1)$, $u^{(3)}_5=(-1, -1, -1)$, $u^{(3)}_6=(1, -1, -1)$,

$u_7^{(3)}=(1, 1, -1)$, $u_8^{(3)}=(-1, 1, -1)$; $u_1^{(4)}=(2, 0, 0)$, $u_2^{(4)}=(0, 2, 0)$,
$u_3^{(4)}=(0, 0, 2)$, $u_4^{(4)}=(-2, 0, 0)$, $u_5^{(4)}=(0, -2, 0)$, $u_6^{(4)}=(0, 0, -2)$;
$u_1^{(5)}=(2, 1, 1)$, $u_2^{(5)}=(2-1, 1)$, $u_3^{(5)}=(2, -1, -1)$, $u_4^{(5)}=(2, 1, -1)$,
$u_5^{(5)}=(-2, -1, -1)$, $u_6^{(5)}=(-2, 1, -1)$, $u_7^{(5)}=(-2, 1, 1)$,
$u_8^{(5)}=(-2, -1, 1)$, $u_9^{(5)}=(1, 2, 1)$, $u_{10}^{(5)}=(1, 2, -1)$, $u_{11}^{(5)}=(-1, 2, -1)$,
$u_{12}^{(5)}=(-1, 2, 1)$, $u_{13}^{(5)}=(-1, -2, -1)$, $u_{14}^{(5)}=(-1, -2, 1)$, $u_{15}^{(5)}=(1, -2, 1)$,
$u_{16}^{(5)}=(1, -2, -1)$, $u_{17}^{(5)}=(1, 1, 2)$, $u_{18}^{(5)}=(-1, 1, 2)$, $u_{19}^{(5)}=(-1, -1, 2)$,
$u_{20}^{(5)}=(1, -1, 2)$, $u_{21}^{(5)}=(-1, -1, -2)$, $u_{22}^{(5)}=(1, -1, 2)$, $u_{23}^{(5)}=(1, 1, -2)$,
$u_{24}^{(5)}=(-1, 1, -2)$.

We set $M_n=\{u_i^{(n)}; 1\leq i\leq d(n)\}$ for $n=1, \cdots, 5$, where $d(1)=6$, $d(2)=12$, $d(3)=8$, $d(4)=6$, $d(5)=24$. For $n\geq 6$, we define $M_n=\{u_i^{(n)}; 1\leq i\leq d(n)\}$, where $u_i^{(n)}=2u_i^{(n-4)}$ $(1\leq i\leq d(n))$ and $d(n)=d(n-4)$. Let $N_n=M_1\cup\cdots\cup M_n$. Then the dimension of the space of summational invariants is 5 for the model N_n, if $n\geq 2$. This can be proved by induction. The regularity of N_n for $n\geq 2$ is shown by using the argument of Cercignani. The case where $G=T_d$ is proved similarly. The modification needed is to replace only M_3 by the 4-velocity model which corresponds to the vertices of a regular tetrahedron. The proof of Theorem 2 will be given in the forthcoming paper [4].

References

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