7. On Modules with Finite Spanning Dimension

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1. Introduction. Let R be a fixed (not necessarily commutative) ring with unity. Throughout in this Note, we are concerned with unital left R-modules (called simply "modules" in the following) $M, A, H, K, \cdots$. Like in Fleury [1] and Rangaswamy [2], we shall use the following terminology. A submodule A of M is called *small* (in M) if $M=A+H$ for any other submodule H of M implies $M=H$. M has a *finite spanning dimension* (abbr. f.s.d.) if for any strictly decreasing sequence $U_0, U_1, U_2, \cdots$ of submodules of M, there is an integer i such that for every $k \geq i$, U_k is small in M. M is *hollow*, if every proper submodule of M is small in M. Then it is proved ([1], Th. 3.1) that any module M with f.s.d. can be expressed as a finite non-redundant sum of hollow submodules: $M=M_1+\cdots+M_p$, and the number p of summands is independent of the ways of decomposition. This number p is called the *spanning dimension* of M and denoted by $Sd(M)$. It is easily proved that if M has f.s.d. then its homomorphic image M/K (K being a submodule of M) has also f.s.d. and $Sd(M/K) \leq Sd(M)$, but it does not hold in general that any submodule of a module with f.s.d. has f.s.d.

Furthermore, if U, X are submodules of M and $M=U+X$ but $M \neq U+Y$ for any proper submodule Y of X, we say that X is a *supplement* of U in M. The following result is proved as Theorem 4.2 in [1]:

If M has f.s.d. and K is a submodule of M which is a supplement of some submodule in M, then K has f.s.d. and $Sd(K)=Sd(M)-Sd(M/K)$.

The purpose of this Note is to prove the following converse of this result, i.e.

Theorem. *Let M be a module with f.s.d. and H a submodule of M also with f.s.d. If*

$$Sd(M)=Sd(H)+Sd(M/H),$$

then H is a supplement of some submodule in M.

2. Lemmas. We list now the lemmas used in the proof of our Theorem. In what follows, M will always mean a module with f.s.d. with $Sd(M)=p$.

Lemma 1. *A submodule H of M has a supplement K^* in M.*

Proof. This follows from Lemma 2.3 of [1].

Lemma 2. *Let H be a nonzero submodule of M with f.s.d. such that $Sd(M)=Sd(M/H)+Sd(H)$. Then H is not small.*

Proof. Let $f: M \to M/H$ be the canonical epimorphism. Let $M=M_1$

$+\cdots+M_p$ be a non-redundant decomposition of M into sum of hollow submodules M_i, $i=1, 2, \cdots, p$. Then it is easy to see that $f(M_i)$ is a hollow submodule of M/H. As $H\neq 0$, we have $Sd(H)>0$, and so our assumption implies $Sd(M/H)<p$. Thus the decomposition $M/H=f(M_1)+\cdots+f(M_p)$ should be redundant, and we may suppose, say, $M/H=f(M_1)+\cdots+f(M_{p-1})$ in which case we should have $M=M_1+\cdots+M_{p-1}+H$, which shows H cannot be small.

Lemma 3. *Let $M=H+K$, where H, K are two submodules of M and $H\neq 0$. Then:*

(i) *H contains a submodule H' which is a supplement of K in M.*

(ii) *H' is then a supplement of $K\cap H$ in H.*

Proof. (i) is contained in the proof of Lemma 2.3 in [1]. (ii) is proved as follows. As $H\supset H'$, we have

$$H'+(K\cap H)=(H'+K)\cap H=M\cap H=H.$$

Let H'' be a submodule of H' such that $H''+(K\cap H)=H$. Then we have $H''+K=H''+(K\cap H)+K=H+K=M$. And as H' is a supplement of K in M, we have $H''=H'$. Thus H' is a supplement of $K\cap H$ in H.

The following Lemma is contained in the proof of Theorems 4.1 and 4.2 in [1].

Lemma 4. (i) *Let H be a submodule of M and K a supplement of H in M. Then K has f.s.d. and we have $Sd(K)=Sd(M/H)$.*

(ii) *If moreover $Sd(K)=Sd(M)$, then $K=M$.*

(iii) *In the same situation as above, let a submodule H' of H be a supplement of K in M. Then K is also a supplement of H' in M.*

(iv) *Let H, K be submodules of M, which are mutually supplement of the other. Then H, K have f.s.d. and we have $Sd(M)=Sd(H)+Sd(K)$.*

3. Proof of the Theorem. We may obviously assume $H\neq 0$. The Lemma 1 assures that H has a supplement K^* in M and our Lemma 2 shows that H is not small. Thus we have $K^*\subsetneq M$.

From Lemma 4 (i), we have

$$Sd(M/H)=Sd(K^*) \tag{1}$$

and from Lemma 3 (i), H contains a submodule H' which is a supplement of K^* in M, so that by Lemma 4 (iii), (iv) we have

$$Sd(M)=Sd(H')+Sd(K^*). \tag{2}$$

From (1), (2) and our assumption, we obtain $Sd(H)=Sd(H')$. H' is, by Lemma 3 (ii), a supplement of $K^*\cap H$ in H. So we have $H=H'$ by Lemma 4 (ii), which shows that H is itself a supplement of K^* in M.

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References

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