32. A Functional Integrating Futaki's Invariant

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This is an announcement of our result generalizing Futaki's invariant on the existence of Einstein Kaehler metrics. Let X be an n-dimensional compact complex connected manifold with ample anti-canonical bundle. Let

$$K := \{\omega \mid \text{Kaehler form on } X \text{ which represents } 2\pi c_1(X)\}.$$

For each element $\omega=\sqrt{-1}\sum g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$ of K, we denote by $\sum R(\omega)_{\alpha\bar\beta}dz^\alpha\otimes d\bar z^\beta$ the corresponding Ricci tensor. Then

$$R(\omega)_{\alpha\bar\beta}=-(\partial^2/\partial z^\alpha\partial\bar z^\beta)(\log(\det(g_{\alpha\bar\beta}))).$$

We put

$$R(\omega) := (\sqrt{-1}/2\pi)\sum R(\omega)_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta.$$

Furthermore, let $\sigma(\omega)$ be the corresponding scalar curvature:

$$\sigma(\omega) := \sum g^{\bar\beta\alpha}R(\omega)_{\alpha\bar\beta},$$

where $(g^{\bar\beta\alpha})$ is the inverse matrix of $(g_{\alpha\bar\beta})$. Fix an arbitrary element ω_0 of K and define a real valued C^∞ function $F_0\in C^\infty(X)_{\boldsymbol{R}}$ on X by

$$2\pi R(\omega_0)-\omega_0=\sqrt{-1}\partial\bar\partial F_0.$$

For each $\varphi\in C^\infty(X)_{\boldsymbol{R}}$, we put $\omega(\varphi) := \omega_0+\sqrt{-1}\partial\bar\partial\varphi$, and let

$$H := \{\varphi\in C^\infty(X)_{\boldsymbol{R}} \mid \omega(\varphi)\in K\}.$$

Note that the natural map

$$\begin{aligned} H &\longrightarrow K \\ \varphi &\longmapsto \omega(\varphi) \end{aligned}$$

is surjective. For every pair $(\omega_1,\omega_2)\in K\times K$, we now define a real number $M(\omega_1,\omega_2)$ by

$$M(\omega_1,\omega_2) := -\int_a^b\left\{\int_X \dot\varphi_t(\sigma(\omega(\varphi_t))-n)\,\omega(\varphi_t)^n\right\}dt,$$

where $\{\varphi_t \mid a\leqq t\leqq b\}$ is an arbitrary piecewise smooth path in H such that $\omega_1=\omega(\varphi_a)$ and $\omega_2=\omega(\varphi_b)$. (Here $\dot\varphi_t$ of course denotes $(\partial/\partial t)(\varphi_t)$.) Then we have

Theorem 1 (Mabuchi [3]). *$M(\omega_1,\omega_2)$ above is independent of the choice of the path $\{\varphi_t \mid a\leqq t\leqq b\}$, and is therefore well-defined. Furthermore $M(\ ,\ )$ satisfies the following cocycle conditions:*

(1) $M(\omega_1,\omega_2)+M(\omega_2,\omega_3)+M(\omega_3,\omega_1)=0,$

(2) $M(\omega_1,\omega_2)+M(\omega_2,\omega_1)=0,$

for all $\omega_1,\omega_2,\omega_3\in K$.

Theorem 2 (Mabuchi [3]). *Let $\mu: K\to\boldsymbol{R}$ be the functional defined by $\mu(\omega)=M(\omega_0,\omega)$ for all $\omega\in K$. Then $\omega=\omega_1$ is a critical point of μ if and only if ω_1 is Einstein Kaehler.*

Theorem 3 (Mabuchi [3]). *Let $\{\psi_t \mid a \leqq t \leqq b\}$ be an arbitrary smooth path in H. We write the corresponding $\omega(\psi_t)$ as*

$$\omega(\psi_t)=\sqrt{-1}\sum g(\psi_t)_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$$

in terms of local coordinates. Let $(g(\psi_t)^{\bar\beta\alpha})$ be the inverse matrix of $(g(\psi_t)_{\alpha\bar\beta})$ and we define a vector field V_t on X by

$$V_t:=\sum g(\psi_t)^{\bar\beta\alpha}(\partial/\partial \bar z^\beta)(\dot\psi_t)\partial/\partial z^\alpha.$$

Then, for every t,

$$(d^2/dt^2)(\mu(\omega(\psi_t)))-(\bar\partial V_t, \bar\partial V_t)_{L^2(X,\omega(\psi_t))}$$
$$=-\int_X \{\ddot\psi_t-\sum g(\psi_t)^{\bar\beta\alpha}\dot\psi_{t;\alpha}\dot\psi_{t;\bar\beta}\}(\sigma(\omega(\psi_t))-n)\omega(\psi_t)^n,$$

where $\dot\psi_{t;\alpha}=(\partial/\partial z^\alpha)(\dot\psi_t)$, $\dot\psi_{t;\bar\beta}=(\partial/\partial \bar z^\beta)(\dot\psi_t)$, and $\ddot\psi_t=(\partial^2/\partial t^2)(\psi_t)$. In particular, if $\tilde\omega$ is a critical point of μ, then for every smooth path $\{\omega_t \mid a \leqq t \leqq a+\varepsilon\}$ in K such that $\omega_a=\tilde\omega$, we have

$$(d^2/dt^2)|_{t=a}(\mu(\omega_t))\geqq 0.$$

Theorem 4 (Mabuchi [3]). *Let Y be an arbitrary holomorphic vector field on X. Let $Y_{\boldsymbol{R}}:=Y+\bar Y$ be the corresponding real vector field and $y_t:=\exp(tY_{\boldsymbol{R}})$ be its 1-parameter group on X. Then, at every $t\in \boldsymbol{R}$ and for all $\omega\in K$,*

$$(d/dt)(\mu(y_t^*(\omega)))=\int_X (Y_{\boldsymbol{R}}F_0)\omega_0^n=2\int_X (YF_0)\omega_0^n.$$

In particular, $\mu(y_t^*(\omega))$ linearly depends on t.

In view of Theorems 2 and 4, one can easily see that if $\int_X (YF_0)\omega_0^n\neq 0$ for some holomorphic vector field Y on X, then μ cannot have a critical point, i.e., X does not admit any Einstein Kaehler metric, which gives another proof of a fundamental theorem of Futaki [2].

Interesting applications and several other generalizations of $M(\ ,\ )$ above will also be given in a forthcoming paper (cf. Bando-Mabucni [1]).

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References

[1] S. Bando and T. Mabuchi: Uniqueness of Einstein Kaehler metrics modulo connected group actions (to appear).
[2] A. Futaki: An obstruction to the existence of Einstein Kaehler metrics. Invent. Math., **73**, 437–443 (1983).
[3] T. Mabuchi: Momentum maps integrating Futaki invariants (to appear).