27. Mixed Problem for Weakly Hyperbolic Equations of Second Order with Degenerate First Order Boundary Condition

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Introduction. In this paper, we are concerned with a mixed problem for second order hyperbolic equations degenerating on the initial surface with degenerate first order boundary condition in $(0, T)\times\Omega$ and prove the existence and uniqueness theorem for classical solutions. The point of our proof is to derive the energy estimate. To do so, we reduce our mixed problem to the one with positive boundary condition for symmetric hyperbolic pseudo differential systems of first order (see [1], [4], [5]). The detailed proof will be given in Tokyo J. Math.

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Writing this paper, we have been informed the result of A. Kubo "On the mixed problems for a weakly hyperbolic equations of second order", which is obtained independently of our paper. Both the result obtained and the method used by him are different from ours.

§1. Statement of the problem and the result. In this paper, we consider the following problem

$$(1.1)\quad \begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2t^k\sum\limits_{j=1}^n h_j(t,x)\dfrac{\partial^2 u}{\partial t\partial x_j}-t^{2k}\sum\limits_{i,j=1}^n a_{ij}(t,x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ \qquad +a_0(t,x)\dfrac{\partial u}{\partial t}+t^{k-1}\sum\limits_{j=1}^n a_j(t,x)\dfrac{\partial u}{\partial x_j}+d(t,x)u=f(t,x) \\ u(0,x)=u_0(x), \qquad u_t(0,x)=u_1(x) \\ B[u]|_S=t^k\Big(\sqrt{\sum\limits_{i,j=1}^n a_{ij}(t,s)\nu_i(s)\nu_j(s)}\dfrac{\partial u}{\partial \nu}+\sum\limits_{j=1}^n \alpha_j(t,s)\dfrac{\partial u}{\partial x_j} \\ \qquad -\beta(t,s)\dfrac{\partial u}{\partial t}+\gamma(t,s)u|_S=g(t,s) \end{cases}$$

in the domain $(0, T)\times\Omega$ where Ω is a bounded domain with smooth boundary $\partial\Omega=S$ in $\boldsymbol{R}^n$, k is a positive integer, $\nu(s)=(\nu_1(s), \cdots, \nu_n(s))$ is the inner unit normal at $s\in S$ and $\nu\cdot\alpha=0$. We assume that all the coefficients belong to $\mathscr{B}([0, T]\times\bar{\Omega})$ or $\mathscr{B}([0, T]\times S)$.

For any $s_0\in S$, there is a following smooth coordinate transformation $\Psi: V\to W$ such that

(i) $\Psi(s_0)=y_0=(0, y_0')=(0, y_{02}, \cdots, y_{0n})$.

(ii) V and W are neighborhoods of s_0 and y_0 respectively.

(iii) $\Psi: V\to W$ is a bijection.

(iv) $\Psi(V\cap \Omega)=W\cap \boldsymbol{R}^n_+$, $\quad \boldsymbol{R}^n_+=\{y=(y_1, \cdots, y_n)\,|\,y_1>0\}$.

(v) $\Psi(V\cap S)=W\cap \boldsymbol{R}^{n-1}$.

and

(vi) L is transformed into the $\tilde{L}$ where

$$\tilde{L}=\frac{\partial^2}{\partial t^2}-2t^k\sum_{j=1}^{n}\tilde{h}_j(t, y)\frac{\partial^2}{\partial t\partial y_j}-t^{2k}\sum_{i,j=1}^{n}\tilde{a}_{ij}(t, y)\frac{\partial^2}{\partial y_i\partial y_j}+\tilde{a}_0(t, y)\frac{\partial}{\partial t}+t^{k-1}\sum_{j=1}^{n}\tilde{a}_j(t, y)\frac{\partial}{\partial y_j}+\tilde{d}(t, y) \tag{1.2}$$

for any $y\in W\cap \bar{\boldsymbol{R}}^n_+$.

We assume the following conditions for the problem (1.1):

(A. I) $\quad L_0\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)=\frac{\partial^2}{\partial t^2}-2\sum_{j=1}^{n}h_j(t, x)\frac{\partial^2}{\partial t\partial x_j}-\sum_{i,j=1}^{n}a_{ij}(t, x)\frac{\partial^2}{\partial x_i\partial x_j}$

is regularly hyperbolic on $[0, T]\times\bar{\Omega}$ and $\sum_{i,j=1}^{n}a_{ij}(t, s)\nu_i(s)\nu_j(s)>0$.

(A. II) (i) By the above coordinate transformation Ψ, B is transformed into $\tilde{B}$ where

$$\begin{aligned}\tilde{B}=&\frac{1}{\sqrt{\tilde{a}_{11}(t, 0, y')}}\left\{t^k\left[\tilde{a}_{11}(t, 0, y')\frac{\partial}{\partial y_1}+\sum_{j=2}^{n}\tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j}\right]\right.\\&\left.+\tilde{h}_1(t, 0, y')\frac{\partial}{\partial t}\right\}+t^k\sum_{j=2}^{n}\tilde{\alpha}_j(t, y')\frac{\partial}{\partial y_j}\\&-\tilde{\beta}(t, y')\left(1+\frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{1/2}\left\{\frac{\partial}{\partial t}-\left(1+\frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{-1}\right.\\&\left.\cdot t^k\sum_{j=2}^{n}\left[\tilde{h}_j(t, 0, y')\frac{\partial}{\partial y_j}-\frac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j}\right]\right\}+\tilde{\gamma}(t, y')\end{aligned} \tag{1.3}$$

for any $y=(0, y')\in W\cap \boldsymbol{R}^{n-1}$.

(ii) The quadratic equation $(c+1)z^2+2bz+(c-1)=0$ has roots in $D=\{z\in \boldsymbol{C}\,|\,|z|<1\}$ where

$$\left\{\begin{aligned}&b=\sum_{j=2}^{n}\tilde{\alpha}_j(t, y')\eta_j/d(\eta'), \qquad c=\tilde{\beta}(t, y')\\&d(\eta')=\left[\sum_{i,j=2}^{n}\tilde{a}_{ij}(t, 0, y')\eta_i\eta_j-\frac{1}{\tilde{a}_{11}(t, 0, y')}\left(\sum_{j=2}^{n}\tilde{a}_{1j}(t, 0, y')\eta_j\right)^2\right.\\&\quad+\left(1+\frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{-1}\cdot\left(\sum_{j=2}^{n}\tilde{h}_j(t, 0, y')\eta_j\right.\\&\quad\left.\left.-\frac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\sum_{j=2}^{n}\tilde{a}_{1j}(t, 0, y')\eta_j\right)^2\right]^{1/2}\end{aligned}\right. \tag{1.4}$$

for any $y=(0, y')\in W\cap \boldsymbol{R}^{n-1}$ and any $\eta'=(\eta_2, \cdots, \eta_n)$.

Definition. We say that the data $\{u_0(x), u_1(x), f(t, x), g(t, s)\}$ satisfy the compatibility condition of infinite order provided

$$\left(\frac{\partial}{\partial t}\right)^r(B[u]|_S)|_{t=0}=\left(\frac{\partial}{\partial t}\right)^r g|_{t=0} \quad (r=0, 1, 2, \cdots).$$

Now we have

Theorem. *For any data* $u_0(x),\ u_1(x)\in H^\infty(\Omega)$, $f(t,x)\in\mathcal{E}_t^\infty(H^\infty(\Omega))$ *and* $g(t,s)\in\mathcal{E}_t^\infty(H^\infty(S))$ *which satisfy the compatibility condition of infinite order, there is a unique solution* $u(t,x)$ *of the problem* (1.1) *which belongs to* $\mathcal{E}_t^\infty(H^\infty(\Omega))$ *where* $H^\infty(\Omega)=\bigcap_{m=0}^{\infty}H^m(\Omega)$.

§2. **Boundary condition.** In this section, we explain the sufficient condition for the existence of the local coordinate transformation Ψ satisfying the assumption (A. II).

We set

$$
(2.1)\quad \begin{cases} L_0\left(t,x,\dfrac{\partial}{\partial t},\dfrac{\partial}{\partial x}\right)=\dfrac{\partial^2}{\partial t^2}-2\sum\limits_{j=1}^{n}h_j(t,x)\dfrac{\partial^2}{\partial t\partial x_j}-\sum\limits_{i,j=1}^{n}a_{ij}(t,x)\dfrac{\partial^2}{\partial x_i\partial x_j} \\ B_0\left(t,s,\dfrac{\partial}{\partial t},\dfrac{\partial}{\partial x}\right)=\sqrt{\sum\limits_{i,j=1}^{n}a_{ij}(t,s)\nu_i(s)\nu_j(s)}\dfrac{\partial}{\partial \nu} \\ \qquad\qquad +\sum\limits_{j=1}^{n}\alpha_j(t,s)\dfrac{\partial}{\partial x_j}-\beta(t,s)\dfrac{\partial}{\partial t} \end{cases}
$$

where $\nu\cdot\alpha=\sum_{j=1}^{n}\nu_j(s)\alpha_j(t,s)=0$. We consider the uniquely determined relation $B_0(t,s,\tau,\xi\nu(s)+\eta)=\tilde{\xi}+b_1\tilde{d}(\eta)-c_1\tilde{\tau}$ for

$$
(2.2)\quad \begin{cases} \tilde{\xi}=-\dfrac{1}{2\sqrt{\sum\limits_{i,j=1}^{n}a_{ij}(t,s)\nu_i(s)\nu_j(s)}}\dfrac{\partial}{\partial \xi}L_0(t,s,\tau,\xi\nu(s)+\eta) \\ \tilde{\tau}=\dfrac{1}{2}\left(1+\dfrac{\left(\sum\limits_{j=1}^{n}h_j(t,s)\nu_j(s)\right)^2}{\sum\limits_{i,j=1}^{n}a_{ij}(t,s)\nu_i(s)\nu_j(s)}\right)^{-1/2}\dfrac{\partial}{\partial\tau}\{L_0(t,s,\tau,\xi\nu(s)+\eta) \\ \qquad\qquad +\tilde{\xi}(t,s,\tau,\xi\nu(s)+\eta)^2\} \\ \tilde{d}(\eta)=\sqrt{\tilde{\tau}^2-\tilde{\xi}^2-L_0(t,s,\tau,\xi\nu(s)+\eta)} \end{cases}
$$

where any $\xi\in\boldsymbol{R}$ and any $\eta\in\boldsymbol{R}^n$ satisfying $\sum_{j=1}^{n}\nu_j(s)\eta_j=0$.

Lemma. *If, for any fixed* $s\in S$, *the quadratic equation* $(c_1+1)z^2+2b_1z+(c_1-1)=0$ *has roots in* $D=\{z\in\boldsymbol{C}\,|\,|z|<1\}$, *we can obtain the local coordinate transformation* Ψ *satisfying* (A. II) *where* b_1 *and* c_1 *are the same in* $B_0(t,s,\tau,\xi\nu(s)+\eta)=\tilde{\xi}+b_1\tilde{d}(\eta)-c_1\tilde{\tau}$.

Proof. We fix $s_0\in S$. Let $\mu=(\mu_1,\cdots,\mu_n)$ be the inner unit normal at s_0. Without loss of generality, we may assume that there exists a smooth function $\rho(x)$ which satisfies the following condition $\rho(V_0\cap S)=0$ and $(\partial\rho/\partial x_1)(V_0)\neq 0$ where V_0 is a neighborhood of s_0. Now we choose $y_1=\rho(x)$ and $y_j=x_j-(\mu_j/\mu_1)x_1$ $(2\leqq j\leqq n)$ as Ψ. Then the condition (A. II) holds (see [3]). Q.E.D.

§3. **Sketch of the proof of Theorem.** First, we mention two facts necessary to obtain Theorem.

By using a partition of unity on $\bar{\Omega}$ and the local coordinate transformation, we have only to treat the mixed problem in $(0,T)\times\boldsymbol{R}_+^n$ and the Cauchy problem in $(0, T)\times \boldsymbol{R}^n$.

For the operator $\tilde{L}$ in (1.2), we have the principal symbol $\sigma_0(\tilde{L}) =\tilde{\tilde{\xi}}^2+\tilde{\tilde{d}}(\eta')^2-\tilde{\tilde{\tau}}^2$ which corresponds to the symbol of the wave equation where

$$
(3.1)\quad \begin{cases}
\tilde{\tilde{\tau}}=\left(1+\dfrac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)\Big\{\tau-\left(1+\dfrac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1} \\
\qquad\qquad \cdot t^k \displaystyle\sum_{j=2}^{n}\left[\tilde{h}_j(t, y)-\dfrac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)}\tilde{a}_{1j}(t, y)\right]\eta_j\Big\} \\
\tilde{\tilde{\xi}}=\dfrac{1}{\sqrt{\tilde{a}_{11}(t, y)}}\Big\{t^k\Big[\tilde{a}_{11}(t, y)\xi+\displaystyle\sum_{j=2}^{n}\tilde{a}_{1j}(t, y)\eta_j\Big]+\tilde{h}_1(t, y)\tau\Big\} \\
\tilde{\tilde{d}}(\eta')=t^k\Big[\displaystyle\sum_{i,j=2}^{n}\tilde{a}_{ij}(t, y)\eta_i\eta_j-\dfrac{1}{\tilde{a}_{11}(t, y)}\Big(\sum_{j=2}^{n}\tilde{a}_{1j}(t, y)\eta_j\Big)^2 \\
\qquad +\left(1+\dfrac{\tilde{h}_1(t, y)^2}{\tilde{a}_{11}(t, y)}\right)^{-1}\cdot\Big(\displaystyle\sum_{j=2}^{n}\Big[\tilde{h}_j(t, y)-\dfrac{\tilde{h}_1(t, y)}{\tilde{a}_{11}(t, y)}\tilde{a}_{ij}(t, y)\Big]\eta_j\Big)^2\Big]^{1/2}
\end{cases}
$$

for $(\xi, \eta')=(\xi, \eta_2, \cdots, \eta_n)\in \boldsymbol{R}^n$ (see [2]). We have $t^k d(\eta')=\tilde{\tilde{d}}(\eta')|_{y_1=0}$ and $\sigma(\tilde{B})=i[\tilde{\tilde{\xi}}+b\tilde{\tilde{d}}(\eta')|_{y_1=0}-c\tilde{\tilde{\tau}}]+\tilde{\gamma}(t, y')$ for $\tilde{B}$ in (1.3), and b and c in (1.4). Also, we have the similar representation when we treat the Cauchy problem (see [2]).

Using (A. I), (A. II), the facts mentioned above and the method to obtain the energy estimate in [5], we consider the localized problem of (1.1) and can obtain Theorem.

References

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