3. *The Exponential Calculus of Microdifferential Operators of Infinite Order.* V

By Takashi AOKI
Department of Mathematics, Kinki University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1984)

1. Introduction. The purpose of this note is to establish a relation between operators with exponential symbols and exponential operators. For each formal symbol q of order $1-0$ (see [1]–[3] for the notation), we construct a formal symbol p of order $1-0$ satisfying

(1.1) $$\exp : p := : \exp q :.$$

2. Statement of the results. We use the same notation as in [2]. Let $q(t; x, \xi)=\sum_{j=0}^{\infty} t^j q_j(x, \xi)$ be a formal symbol of order at most $1-0$ defined in a conic open set Ω in T^*X. We introduce inductively a sequence of symbols $\{\psi_{l,k}^{(j)}(x, y, \xi, \eta)\}$ defined in $\Omega \times \Omega$ by the following:

(2.1) $$\psi_{0,0}^{(0)}(x, y, \xi, \eta)=q_0(x, \xi),$$

(2.2) $$\psi_{l,0}^{(j)}(x, y, \xi, \eta)=0, \qquad j=1, 2, \cdots ; l=0, 1, 2, \cdots,$$

(2.3) $$\psi_{l,k+1}^{(j)}(x, y, \xi, \eta)=\frac{1}{k+1}\Big\{\partial_\xi \cdot \partial_y \psi_{l,k}^{(j)}(x, y, \xi, \eta) + \sum_{\nu=0}^{l} \sum_{\mu=0}^{j-1} \sum_{i=0}^{l-\nu} \frac{1}{j-\mu} \partial_\xi \psi_{\nu,k}^{(\mu)}(x, y, \xi, \eta) \cdot \partial_y \psi_{i,l-\nu-i}^{(j-\mu-1)}(y, y, \eta, \eta)\Big\},$$

(2.4) $$\psi_{k,0}^{(0)}(x, y, \xi, \eta)=q_k(x, \xi)-\sum_{j=0}^{k} \sum_{l=0}^{k-1} \frac{1}{j+1} \psi_{l,k-l}^{(j)}(x, x, \xi, \xi).$$

If $\psi_{l,k}^{(j)}$ is known for $l+k \le m$, then $\psi_{l,k}^{(j)}$ is defined for $l+k \le m+1$ by (2.2)–(2.4). We set

(2.5) $$p_k(x, \xi)=\psi_{k,0}^{(0)}(x, x, \xi, \xi)$$

and define a formal power series in t by

(2.6) $$p(t; x, \xi)=\sum_{k=0}^{\infty} t^k p_k(x, \xi).$$

Remark. $\psi_{k,0}^{(0)}$ is independent of (y, η).

Theorem 1. *The formal series $p(t; x, \xi)$ is a formal symbol of order at most $1-0$ defined in Ω so that*

(2.7) $$\exp : p(t; x, \xi) := : \exp q(t; x, \xi) :$$

holds in $\mathcal{E}^R$.

Let λ be a real number such that $0 \le \lambda < 1$. Then we have the following

Theorem 2. *If $q_j(x, \xi)$ is of order at most $(j+1)\lambda - j$ $(j=0, 1, 2, \cdots)$, then $p_k(x, \xi)$ is of order at most $(k+1)\lambda - k$ $(k=0, 1, 2, \cdots)$.*

3. Invertibility. As an application of Theorem 1, we obtain the following theorem of invertibility for operators of infinite order, which is a generalization of Theorem 5 in [1]:

Theorem 3. *Let $P(x,\xi)$ be a symbol defined in a conic open neighborhood of $\mathring{x}^* \in T^*X$. If $P(x,\xi)$ is invertible as a symbol, that is, $1/P(x,\xi)$ is also a symbol defined near $\mathring{x}^*$, then $:P(x,\xi):$ is invertible in $\mathcal{E}^R_{\mathring{x}^*}$.*

4. Outline of the proof of Theorem 1. It follows from Theorem 2 in [2] that, if $p(t;x,\xi)=\sum t^j p_j(x,\xi)$ is given, then $q(t;x,\xi)$ satisfying (2.7) is constructed as follows: Let us introduce $\{\psi^{(j)}_{l,k}(x,y,\xi,\eta)\}$ and $\{q^{(j)}_k(x,\xi)\}$ by

$$\psi^{(0)}_{l,0}=p_l(x,\xi), \qquad l=0,1,2,\cdots, \tag{4.1}$$

$$\psi^{(j)}_{l,0}=0, \quad j=1,2,\cdots; \quad l=0,1,2,\cdots, \tag{4.2}$$

$$q^{(j+1)}_k(x,\xi)=\frac{1}{j+1}\sum_{l=0}^{k}\psi^{(j)}_{l,k-l}(x,x,\xi,\xi), \tag{4.3}$$

$$\psi^{(j)}_{l,k+1}=\frac{1}{k+1}\left(\partial_\xi\cdot\partial_y\psi^{(j)}_{l,k}+\sum_{\nu=0}^{l}\sum_{\mu=0}^{j-1}\partial_\xi\psi^{(\mu)}_{\nu,k}\cdot\partial_y q^{(j-\mu)}_{l-\nu}(y,\eta)\right). \tag{4.4}$$

Then $q(t;x,\xi)=\sum_{k=0}^{\infty}t^k q_k(x,\xi)$ is obtained by

$$q_k(x,\xi)=\sum_{j=1}^{k+1}q^{(j)}_k(x,\xi). \tag{4.5}$$

If we can solve $\{p_k(x,\xi)\}$ from $\{q_j(x,\xi)\}$ conversely by (4.1)–(4.5), then $p(t;x,\xi)=\sum t^k p_k(x,\xi)$ satisfies, at least formally, (2.7) for given $q(t;x,\xi)=\sum t^j q_j(x,\xi)$. Such procedure can be done by eliminating $q^{(j)}_k$'s from (4.3)–(4.5). Then we have (2.1)–(2.5).

References

[1] T. Aoki: The exponential calculus of microdifferential operators of infinite order. II. Proc. Japan Acad., **58**A, 154–157 (1982).

[2] ——: ditto. IV. ibid., **59**A, 186–187 (1983).

[3] ——: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini, I (to appear in Ann. Inst. Fourier, Grenoble, **33**-4, (1983)).