20. *The Support of Global Graph Links*[*)]

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Introduction. In [7], [8], Tamura initiated *global knot theory.* Knots are, by definition, codimension two spheres embedded in manifolds. Tamura calls a knot *local* if it is contained in an embedded ball in the ambient manifold and *unknotted* if it bounds an embedded disk. The following is one of the fundamental problems in global knot theory: *Give criteria for a knot to be local or to be unknotted.* In this note, we study this problem for knots and links in three dimensional manifolds. To avoid the Poincaré conjecture, we need the notion of quasi-localness: A knot is *quasi-local* if it is contained in a homotopy ball embedded in the ambient manifold. Then the fundamental group of the knot exterior determines the quasi-localness and the unknottedness (§ 4).

In contrast with this, if one looks for a criterion for the localness or the quasi-localness of a knot in terms of its homotopy class in the ambient manifold, one encounters the following difficulty: Local knots are *inessential* i.e., null homotopic in the ambient manifold, but the converse does not hold.

In this note we show, however, that if we restrict ourselves to certain graph knots and graph links, the converse does hold. For the precise argument, we introduce the notion of the support of a knot or a link in an orientable three dimensional manifold (§ 1). We then give all the possibility of the support of an inessential graph link (§ 2). Moreover we show that under certain condition on the ambient manifold, the support of a graph link is determined by the homotopy class of each component (§ 3).

This note is an announcement of the author's doctoral thesis at the University of Tokyo. Details will appear elsewhere.

1. Definitions. Manifolds are assumed to be connected, compact, oriented, and of dimension three and links are oriented and may be empty. We say that a manifold is *prime to* $S^1 \times S^2$ if its prime decomposition does not contain $S^1 \times S^2$. A link is *inessential* if each component of it is null homotopic in the ambient manifold and is *nonsplittable* if there does not exist an embedded S^2 in the ambient manifold which separates it into two nonempty sublinks.

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Definition. Let L be a link in a manifold M. When L is nonsplittable, there exist a link L_0 in a manifold M_0 with irreducible exterior and another manifold M_1 such that the pair (M, L) is diffeomorphic to $(M_0 \# M_1, L_0)$. Then the *support* of L is the manifold M_0. In general, L is decomposed into nonsplittable sublinks $L=\bigcup L_j$, and then the *support* of L is the connected sum of the supports of L_j's. Then a link is local and quasi-local if its support is S^3 and a homotopy sphere, respectively. We say that a link is *full* if the ambient manifold coincides with its support.

A manifold M is a *graph manifold* if there is a family of disjointly embedded tori in M such that each connected component of the manifold obtained by cutting M along these tori is the total space of an S^1-bundle over a surface. A link is a *graph link* if its exterior is a graph manifold. We need the notion of $\mathcal{R}$-decompositions of graph manifolds, which is a modification of that of round handle decompositions due to Asimov [1] (see also Morgan [4]).

Definition. A decomposition $\mathcal{D}=\{S_1, \cdots, S_m ; H_1, \cdots, H_n\}$ of a manifold M is called an *$\mathcal{R}$-decomposition* if

(1) there is a family of disjointly embedded tori such that $\mathcal{D}$ is the set of connected components of the manifold obtained by cutting M along these tori, and

(2) $S_i \simeq S^1 \times D^2 \quad i=1, \cdots, m$, and
$H_j \simeq S^1 \times$(two punctured disk) $\quad j=1, \cdots, n$.

An $\mathcal{R}$-decomposition $\mathcal{D}$ of M is called *minimal* if none of unions of more than one elements of $\mathcal{D}$ in M is again diffeomorphic either to $S^1 \times D^2$ or to $S^1 \times$(two punctured disk).

2. Characterization of graph links and main results. For an $\mathcal{R}$-decomposition $\mathcal{D}=\{S_1, \cdots, S_m ; H_1, \cdots, H_n\}$ of M, we let $\partial\mathcal{D}$ denote the family of disjointly embedded tori

$$\{\text{connected components of } \partial H_j\}_{j=1,\dots,n} - \{\partial S_i\}_{i=1,\dots,m}.$$

We say that a link $L=\bigcup_{i=1}^{l} k_i$ in M *lies on* $T \in \partial\mathcal{D}$, if there is a collar $T \times [0,1]$ of T in M and k_i is a knot on $T \times \{i/l\}$ not null homotopic in $T \times \{i/l\}$.

Definition. A graph link L in M is *elementary* if the pair (M, L) satisfies one of the following:

(1) M is a graph manifold and L is empty,

(2) M is $S^1 \times S^2$ and L is given by $L=S^1 \times \{*\}$, or

(3) M is an irreducible graph manifold and $\mathcal{D}=\{S_1, \cdots, S_m ; H_1, \cdots, H_n\}$ is some minimal $\mathcal{R}$-decomposition of M with $\partial\mathcal{D} =\{T_1, \cdots, T_l\}$. For subsets I' of $\{1, \cdots, m\}$ and J' of $\{1, \cdots, l\}$, the link L is given by $L=\bigcup_{j' \in J'} L_{j'} \cup \bigcup_{i' \in I'} k_{i'}$, where the sublink $L_{j'}$ lies on $T_{j'} \in \partial\mathcal{D}$ and the knot $k_{i'}$ is the core of $S_{i'} \in \mathcal{D}$.

Then we have a constructive characterization of global links.

Theorem. *Every graph link is obtained from elementary ones by a finite number of operations of disjoint union (the ambient manifold is the connected sum), adding a cable knot of some component, replacing one component by its cable knot, link connected sum, and self link connected sum.*

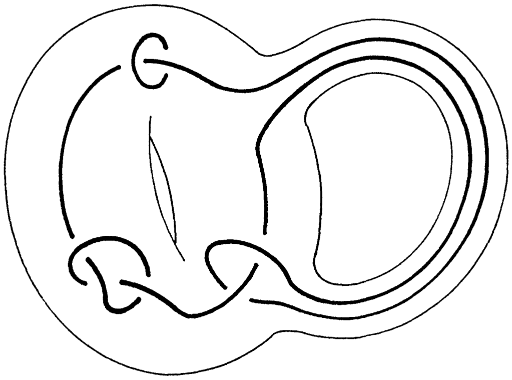

Fig. 1

Self link connected sum is indicated by Fig. 1. This characterization is a generalization of that of graph knots in S^3 due to Gordon [3] and Ue [9] (see also Soma [6]).

The following is proved by induction on the number of operations in the theorem above.

Theorem. *The support of an inessential graph link is a connected sum of $S^1 \times S^2$'s and graph manifolds with finite fundamental groups.*

Corollary. *Suppose that M is a graph manifold prime to $S^1 \times S^2$ and that the fundamental group of M is torsion-free. Then a graph link in M is local if and only if it is inessential.*

We remark that any connected sum of $S^1 \times S^2$'s and graph manifolds with finite fundamental groups admits an inessential full graph knot.

Now it is natural to ask: If one assumes the localness of each component instead of inessentiality, what is the support of a link? For graph links, we have

Theorem. *The support of a graph link consisting of local knots is a connected sum of $S^1 \times S^2$'s and lens spaces.*

3. Homotopy support. Let M be a manifold prime to $S^1 \times S^2$ and $\phi : M \simeq M_1 \# \cdots \# M_k$ be a homeomorphism which gives a prime decomposition of M. Then there is a natural isomorphism $\phi_* : \pi_1(M) \to \pi_1(M_1) * \cdots * \pi_1(M_k)$. Let $p_j : \pi_1(M_1) * \cdots * \pi_1(M_k) \to \pi_1(M_j)$ be a homomorphism given by adding relations $\pi_1(M_i)=1$ for $i \neq j$. Then for a link $L = \bigcup k_i$ in M, we define the *homotopy support* of L to be the connected sum of M_j's such that there is a component k_i with $p_j(\phi_*[k_i]) \neq 1$ in $\pi_1(M_j)$. Then the result in § 2 is generalized as follows:

Theorem. *The support of a graph link in a manifold prime to $S^1 \times S^2$ is a connected sum of its homotopy support and graph manifolds with finite fundamental groups.*

4. Remarks. As noted in the introduction, results in § 2 do not hold for general knots and links. In fact, we know by a result of Bing [2] that any closed manifold admits an inessential full knot as well as a full link which consists of local knots. However, since Kneser's conjecture is true, the fundamental group of the exterior of a knot or a link determines the quasi-localness and the unknottedness.

Quasi-localness theorem for a link. *Let L be a link in a closed manifold M which is prime to $S^1 \times S^2$. Then L is quasi-local if and only if there is a group G such that $\pi_1(E(L)) \simeq \pi_1(M) * G$, where $E(L)$ denotes the exterior of L.*

For knots, the condition on the ambient manifold can be omitted.

Quasi-localness theorem for a knot. *A knot k in a closed manifold M is quasi-local if and only if there is a non-trivial group G such that $\pi_1(E(k)) \simeq \pi_1(M) * G$.*

As a corollary to this, we get a global version of Papakyriakopoulos' unknotting theorem [5].

Unknotting theorem for a knot. *A knot k in a closed manifold M is unknotted if and only if $\pi_1(E(k)) \simeq \pi_1(M) * \boldsymbol{Z}$.*

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