

2. A Generalization of Liapunov's Theorem Concerning a Mass of Fluid with Self-Gravitation^{*}

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§ 1. Introduction. Suppose that a mass of fluid with uniform density lies in \mathbf{R}^3 and that no force other than the gravitational one due to itself acts on it. Then it is intuitively clear that the fluid attains its equilibrium by forming a sphere. M. A. Liapunov proves in [1] that the sphere is the only stable equilibrium figure of the fluid (see also Poincaré [2]). In the present paper we show that among all the figures of the mass of fluid (stable or not) the sphere is the only possible equilibrium figure. Our proof is completely different from Liapunov's. Actually our method is Serrin's moving plane method ([6]).

Remark. We consider only the gravitational force. In particular, the fluid lies still without rotation. In the case where the fluid rotates with a uniform angular velocity, various kinds of equilibrium figures are known to occur to form bifurcations (see [1]–[3], [7], [8]).

§ 2. Mathematical formulation of the problem. Let Ω be the domain occupied by the fluid. Suppose that Ω is a bounded connected open set in \mathbf{R}^3 with a boundary Γ of C^1 -class. The density of the fluid is assumed to be unity. We denote by V the potential of the gravitational force vanishing at the infinity. Then V is given by

$$V(x) = \int_{\Omega} \frac{dy}{|x-y|},$$

if the scales are suitably chosen. The function V is characterized by $V \in C^1(\mathbf{R}^3)$ and

$$\begin{aligned} (1) \quad & -\Delta V = 4\pi \quad \text{in } \Omega, \\ (2) \quad & -\Delta V = 0 \quad \text{in } \mathbf{R}^3 \setminus \Omega, \\ (3) \quad & V(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

The equation of motion is easily integrated to yield $P = V + \text{constant}$, where P is the pressure. Consequently the equilibrium state is represented by

$$(4) \quad V = \text{constant} \quad \text{on } \Gamma.$$

Hence our goal is to show the following

Theorem 1. *If $V \in C^1(\mathbf{R}^3)$ satisfies (1)–(4), then Γ is necessarily a sphere.*

^{*} Partially Supported by the Fûjukai.

§ 3. **Proof.** To show Theorem 1 we use the moving plane method due to Serrin [6]. This method is also used in Gidas, Ni and Nirenberg [4] and Matano [5] to yield fascinating results.

Take an arbitrary direction and choose it as the x -axis. We denote by $T(\xi)$ the plane through $(\xi, 0, 0)$ perpendicular to the x -axis. We put $\xi_0 = \inf \{\xi; T(\eta) \cap \Gamma = \emptyset (\eta > \xi)\}$. For $\xi < \xi_0$ we consider the subdomain of Ω to the right hand side of $T(\xi)$. Then the subdomain is reflected to the left hand side of $T(\xi)$. The reflected domain is denoted by $D(\xi)$. If $\xi < \xi_0$ but is sufficiently close to ξ_0 , then the domain $D(\xi)$ lies inside of Ω . Put $\xi_1 = \inf \{\xi < \xi_0; D(\xi) \subset \Omega\}$. As for $D(\xi_1)$, either the following I) or II) holds true.

I) There is a point $P \in \partial D(\xi_1) \setminus T(\xi_1)$ at which $\partial D(\xi_1)$ is tangent to Γ (see Fig. 1).

II) At some point $P \in T(\xi_1) \cap \partial D(\xi_1) \cap \Gamma$, Γ is tangent to $\partial D(\xi_1)$ (see Fig. 2).

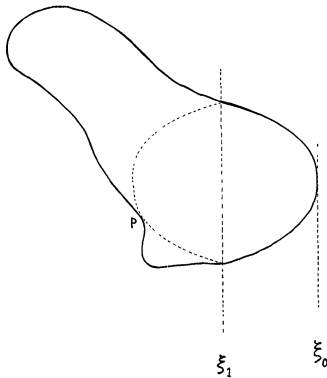


Fig. 1

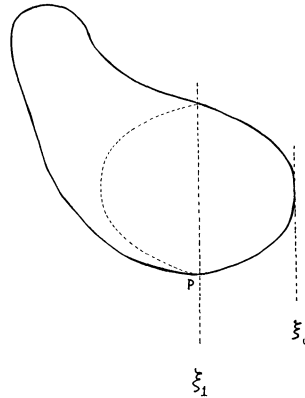


Fig. 2

We will show in both cases that V is symmetric with respect to $T(\xi_1)$.

The case of I). We denote the reflection with respect to $T(\xi_1)$ by $Q \rightarrow Q^*$. Defining a function U by $U(Q) = V(Q^*)$, we consider $W = V - U$ in the half plane to the left of $T(\xi_1)$. Then, as is easily seen, the function W is superharmonic and vanishes at ∞ and on $T(\xi_1)$. On the other hand, $W(P) = V(P) - U(P) = 0$, since V is constant on Γ . Hence we have $W \equiv 0$ by the maximum principle.

The case of II). The function W is superharmonic and vanishes at ∞ and on $T(\xi_1)$. At $P \in T(\xi_1)$ W has the minimum. Therefore by the maximum principle we have either $(\partial W / \partial x)(P) < 0$ or $W \equiv 0$. On the other hand, by the fact that $\partial D(\xi_1)$ is tangent to Γ at P and that V is constant along Γ and U along $\partial D(\xi_1) \setminus T(\xi_1)$, we obtain $(\partial W / \partial x)(P) = 0$. Therefore we have $W \equiv 0$.

By the discussion above we have $W \equiv 0$ in both cases, i.e., the

function V is symmetric with respect to $T(\xi_1)$. This, in turn, implies that Γ must be a sphere, since the choice of the direction of the x -axis is arbitrary.

§ 4. The case in the presence of surface tension. In the preceding discussion we have neglected the effect of the surface tension. In this section we derive the same conclusion under the assumption that the surface tension coefficient σ is a positive constant.

Theorem 2. *Suppose that $V \in C^1(\mathbf{R}^3)$ satisfies (1)–(3) and*

$$V = \sigma H_\Gamma + \text{constant} \quad \text{on } \Gamma.$$

Here H_Γ is the mean curvature of Γ . σ is a positive constant. Suppose moreover that Γ is a surface of C^3 -class. Then Γ is necessarily a sphere.

Proof. The proof is carried out along the same line as the proof of Theorem 1, hence we use the same notation. In both cases of I) and II), $W \equiv V - U$ is superharmonic and vanishes at ∞ and on $T(\xi_1)$.

The case of I). We show that $W(P) \leq 0$. To this end we consider the inward normal through P . We take this as the z -axis and choose a corresponding Cartesian coordinate system x, y, z . Then Γ is represented, near P , by the equation $z = f(x, y)$ with $f \in C^3$ satisfying $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Here the subscript implies the differentiation. On the other hand, $\partial D(\xi_1)$ is represented, near P , by the equation $z = g(x, y)$ with $g \in C^3$, $g(0, 0) = g_x(0, 0) = g_y(0, 0) = 0$. By the definition of P we have $g(x, y) \geq f(x, y)$. We see easily that $2H_\Gamma = \Delta f(0, 0)$, $2H_{\partial D(\xi_1)} = \Delta g(0, 0)$ at P . Hence $W(P) = \sigma H_\Gamma(P) - \sigma H_{\partial D(\xi_1)}(P) = (\sigma/2)\Delta(f - g)(0, 0)$. But the function $h \equiv f - g$ satisfies $h(0, 0) = h_x(0, 0) = h_y(0, 0) = 0$ and $h(x, y) \leq 0$. Consequently $\Delta h(0, 0) \leq 0$, i.e., $W(P) \leq 0$. Now by the maximum principle, we obtain $W \equiv 0$.

The case of II). Take P as the origin and employ a coordinate system with the inward normal at P as the z -axis and $T(\xi_1)$ as the yz -plane. Then Γ is represented near P by the equation $z = f(x, y)$ with $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Calculation shows that $(\partial W / \partial x)(P) = \sigma \Delta f_x(0, 0)$. If we assume that W is not identically zero, then we have $\Delta f_x(0, 0) < 0$ by the maximum principle. On the other hand, by the definition of ξ_1 , we have $f(x, y) \geq f(-x, y)$ for sufficiently small y and nonnegative x . This fact implies that $f_{xxx}(0, 0) + f_{xyy}(0, 0) \geq 0$. Indeed, expanding as

$$f(x, y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta_1 x^3 + \delta_2 x^2 y + \delta_3 x y^2 + \delta_4 y^3$$

$$(\alpha, \beta, \gamma : \text{constants}, \delta_j = \delta_j(x, y) \ (1 \leq j \leq 4)),$$

we first obtain $\beta = 0$ and then, taking $x = \sqrt{3}y$, we obtain $3\delta_1(0, 0) + \delta_3(0, 0) \geq 0$, which implies $f_{xxx}(0, 0) + f_{xyy}(0, 0) \geq 0$. Now we have a contradiction since $\Delta f_x(0, 0) < 0$. Therefore $W \equiv 0$. Then by the same reason as before we can conclude that Γ is a sphere.

References

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