11. *Small Deformations of Certain Compact Class L Manifolds*

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The notion of Class L manifolds was introduced by Ma. Kato [1]. The most significant property of Class L is that any two members of Class L can be connected complex analytically to obtain another Class L manifold. The purpose of this note is to construct a series of compact Class L 3-folds $\{M(n)\}_{n\in N}$ inductively and to determine their all small deformations. Details will be published elsewhere.

1. We denote the 3-dimensional complex projective space by $\boldsymbol{P}^3$ of which the system of homogeneous coordinates we write $[\zeta_0:\zeta_1:\zeta_2:\zeta_3]$. For any positive real number r, we define a domain U_r in $\boldsymbol{P}^3$ by $U_r =\{[\zeta_0:\zeta_1:\zeta_3:\zeta_3]\in \boldsymbol{P}^3 \,|\, |\zeta_0|^2+|\zeta_1|^2<r(|\zeta_2|^2+|\zeta_3|^2)\}$. A complex 3-fold X is said to be of *Class L* if it contains a domain which is biholomorphic to U_1, in other words, if there exists a holomorphic open embedding of U_1 into X. Let σ be a holomorphic automorphism of $\boldsymbol{P}^3$ defined by $\sigma([\zeta_0:\zeta_1:\zeta_2:\zeta_3])=[\zeta_2:\zeta_3:\zeta_0:\zeta_1]$. For any real number ε greater than 1, we denote the domain $U_\varepsilon-\overline{U_{1/\varepsilon}}$ by $N(\varepsilon)$ where $\overline{}$ indicates the topological closure. Then it is easy to see that U_r is isomorphic to $U=U_1$ and that $\sigma(N(\varepsilon))=N(\varepsilon)$.

Suppose that X_1 and X_2 are Class L manifolds with open embeddings $i_\nu: U_\varepsilon\to X_\nu$, $\nu=1, 2$. Put $X_\nu^\sharp=X_\nu-\overline{i_\nu(U_{1/\varepsilon})}$. We define a complex manifold $Z(X_1, X_2, i_1, i_2)=X_1^\sharp\cup X_2^\sharp$ by identifying a point $x_1\in i_1(N(\varepsilon))\subset X_1^\sharp$ with the point $x_2=i_2\circ\sigma\circ i_1^{-1}(x_1)\in X_2^\sharp$. $Z(X_1, X_2, i_1, i_2)$ is also a Class L manifold because $N(\varepsilon)$ is of Class L. Remark that the construction of $Z(X_1, X_2, i_1, i_2)$ depends on the choice of the open embeddings i_1 and i_2.

Now we define a compact Class L manifold $M=M(1)$. Let l_0 and l_∞ be projective lines in $\boldsymbol{P}^3$ given by

$$l_0:\zeta_0=\zeta_1=0, \qquad l_\infty:\zeta_2=\zeta_3=0,$$

and put $W=\boldsymbol{P}^3-l_0-l_\infty$. Consider the holomorphic automorphism g: $[\zeta_0:\zeta_1:\zeta_2:\zeta_3]\mapsto[\zeta_0:\zeta_1:\alpha\zeta_2:\alpha\zeta_3]$ of W, where α is a complex number with $0<|\alpha|<1$. Letting $\langle g\rangle$ be the infinite cyclic group generated by g, we define the complex manifold M to be the quotient space of W by $\langle g\rangle$. Taking real numbers β, γ, δ such that $|\alpha|<\beta<\gamma<\delta<1$, we define subdomains U_0, U_W, U_∞ in W as follows:

$$U_0=U_\delta-\overline{U_{|\alpha|}},\quad U_W=N\Big(\frac{1}{\gamma}\Big),\quad U_\infty=U_{\beta/|\alpha|^2}-\overline{U_{1/\delta}}.$$

By the above definition, we have

$$gU_0\cap U_\infty=U_{\beta/|\alpha|^2}-\overline{U_{1/|\alpha|}}\neq\phi,\qquad gU_W\cap U_\infty=\phi.$$

This shows that M is obtained by identifying $\zeta\in gU_0\cap U_\infty$ with $g^{-1}(\zeta)\in U_0$ in $U_0\cup U_W\cup U_\infty$. As already remarked, $N(1/\gamma)$ is of Class L. Hence M is a Class L manifold.

Let us construct the sequence of compact Class L manifolds $\{M(n)\}_{n\in N}$. Fix a small positive number λ. Let ι_λ be a holomorphic mapping of U_ε into U_W given by $\iota_\lambda([\zeta_0:\zeta_1:\zeta_2:\zeta_3])=[\zeta_0+\lambda\zeta_2:\zeta_1+\lambda\zeta_3:\lambda\zeta_2-\zeta_0:\lambda\zeta_3-\zeta_1]$. We define $M(2)=Z(M,M,\iota_\lambda,\iota_\lambda)$. Suppose that we have $M(n)=Z(M(n-1),M,\iota,\iota_\lambda)$ where $\iota: U_\varepsilon\to M(n-1)$. We define $M(n+1)=Z(M(n),M,\iota'_\lambda,\iota_\lambda)$ where $\iota'_\lambda: U_\varepsilon\to\iota(N(\varepsilon))\subset M(n)$ taking suitable $\lambda'>0$.

2. Small deformations of M. It is easily seen that M is a fibre bundle over $P^1\times P^1$. On the cohomologies of M with coefficient in Θ, we have:

a) $\dim H^0(M,\Theta)=7$, b) $\dim H^1(M,\Theta)=7$,

c) $\dim H^2(M,\Theta)=0$, d) $\dim H^3(M,\Theta)=0$.

First a) is easily shown by calculations. d) is due to [1] p. 12, Proposition 2.3. c) is verified by applying the spectral sequence to the fibre bundle M over $P^1\times P^1$. Since we can see that all the Chern numbers of M vanish, we get d) by the Riemann-Roch theorem and by the results a), c) and d).

We define a complex manifold $\mathcal{M}$ as follows. Let B be a domain in C^7 defined by

$$B=\{t=(t_1,t_2,\cdots,t_7)\in C^7\,|\,|t_i|<\delta(i=1,2,\cdots,7)\}$$

where δ is a sufficiently small positive number. Let g_t be a holomorphic automorphism of W given by $g_t([\zeta_0:\zeta_1:\zeta_2:\zeta_3])=[\zeta_0+t_1\zeta_1:t_2\zeta_0+(1+t_3)\zeta_1:\alpha(1+t_4)\zeta_2+t_5\zeta_3:t_6\zeta_2+\alpha(1+t_7)\zeta_3]$ for $t\in B$. We put $\tilde{g}(\zeta,t)=(g_t(\zeta),t)$, then $\tilde{g}$ is a holomorphic automorphism of $W\times B$. The quotient space of $W\times B$ by $\langle\tilde{g}\rangle$ is a complex manifold. We define $\mathcal{M}$ by $W\times B/\langle\tilde{g}\rangle$. The projection $\varpi: W\times B\to B$ induces a projection of $\mathcal{M}$ to B because ϖ and g commute, i.e., $\varpi\circ g=\varpi$. It is easily checked that $(\mathcal{M},B,\varpi)$ is a complex analytic family with $\varpi^{-1}(0)=M$. We can show that the Kodaira-Spencer map at 0 is an isomorphism.

Theorem 1. *$(\mathcal{M},B,\varpi)$ is the complete, effectively parametrized complex analytic family of the small deformations for M, in other words, any small deformation of M is biholomorphic to $W/\langle g_t\rangle$ for some $t\in B$.*

3. Small deformations of $M(n)(n\geqq 2)$. First we show

Proposition. *Let X_1 and X_2 be compact Class L manifolds. Let*

$X_1 \# X_2$ *denote any manifold obtained by connecting* X_1 *and* X_2. *Then we have*

$$\dim H^2(X_1 \# X_2, \Theta) = \dim H^2(X_1, \Theta) + \dim H^2(X_2, \Theta).$$

This can be proved by using the Mayer-Vietoris exact sequence for $X_1^\# \cup X_2^\#$ and the exact sequences of local cohomologies for the pairs $(X_i, X_i^\#)(i=1,2)$.

On the cohomologies of $M(n)$, $n \geqq 2$, we have

a) $\dim H^0(M(n), \Theta) = 3$,

b) $\dim H^1(M(n), \Theta) = 15n - 12$,

c) $\dim H^2(M(n), \Theta) = 0$,

d) $\dim H^3(M(n), \Theta) = 0$.

a) is shown by easy calculations and d) is due to [1]. c) is shown by the above proposition and the fact that $H^2(M, \Theta) = 0$. b) is proved by the Riemann-Roch theorem and the formula on the Chern numbers of connected sums of Class L manifolds [1] and the results a), c) and d).

Letting δ be a small positive real number and $B(t')$ a domain in $\boldsymbol{C}^4$ defined by

$$B(t') = \{t' = (t_1', t_2', t_3', t_4') \in \boldsymbol{C}^4 \,|\, |t_i'| < \delta \ (i = 1, \cdots, 4)\},$$

we define a holomorphic open embedding $s_{t'}$ of $N(\eta) \subset U_W \subset M$ into U_W by

$$s_{t'}([\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3]) = [\mu\zeta_0 + \nu\zeta_2 : \nu t_1'\zeta_0 + (\mu + \nu t_2')\zeta_1 + \mu t_1'\zeta_2 + (\nu + \mu t_2')\zeta_3 :$$
$$-(\nu(1+t_3')\zeta_0 + \nu t_4'\zeta_1 + \mu(1+t_3')\zeta_2 + \mu t_4'\zeta_3) : -(\nu\zeta_1 + \mu\zeta_3)]$$

where $\mu = 1 + \lambda^2$, $\nu = 1 - \lambda^2$. $s_{t'}$ is well-defined if we take λ and η so small that $\iota_\lambda(N(\varepsilon)) \subset N(\eta) \subset U_W$. We restrict $s_{t'}$ to $(U_W - \overline{\iota_\lambda(N(\varepsilon))}) \cap s_{t'}^{-1}(U_W)$ which we shall denote for simplicity also by $s_{t'}$, then $s_{t'}$ becomes a holomorphic open embedding of $(U_W - \overline{\iota_\lambda(N(\varepsilon))}) \cap s_{t'}^{-1}(U_W)$ into U_W.

Now we construct a complex manifold $\mathcal{M}(2)$ as follows. First take two copies of $\mathcal{M}$. We write (x^1, t^1) a point of one of the copies and (x^2, t^2) that of another to distinguish them from each other. From Theorem 1, $M_t = \varpi^{-1}(t)$ contains U_W and U_W contains $\iota_\lambda(U_{1/\varepsilon})$. Put $\mathcal{M}^\# = \mathcal{M} - (\overline{\iota_\lambda(U_{1/\varepsilon})} \times B)$. We define $\mathcal{M}(2) = \mathcal{M}^\# \times B \times B(t') \cup \mathcal{M}^\# \times B \times B(t')$ by identifying

$$((x^1, t^1),\ t^2, t') \in \iota_\lambda(N(\varepsilon)) \times B \times B \times B(t') \subset \mathcal{M}^\# \times B \times B(t')$$

with

$$((x^2, \tilde{t}^2),\ \tilde{t}^1, \tilde{t}') \in \iota_\lambda(N(\varepsilon)) \times B \times B \times B(t') \subset \mathcal{M}^\# \times B \times B(t')$$

if and only if $x^2 = s_{t'}(x^1)$, $t^1 = \tilde{t}^1$, $t^2 = \tilde{t}^2$, $t' = \tilde{t}'$. We can easily define the projection ϖ of $\mathcal{M}(2)$ to $B \times B \times B(t')$. Then it is clear that $(\mathcal{M}(2), B \times B \times B(t'), \varpi)$ becomes a complex analytic family with $\varpi^{-1}(0) = M(2)$. Studying the Kodaira-Spencer map, we get

Theorem 2. *$(\mathcal{M}(2), B \times B \times B(t'), \varpi)$ is the complete, effectively parametrized complex analytic family of the small deformations of $M(2)$.*

For $n \geqq 3$, we can construct the complete, effectively parametrized complex analytic family of the small deformations of $M(n)$ inductively. Let

$$B(t'')=\{t''=(t''_1, t''_2, \cdots, t''_8) \in C^8 \mid |t''_i|<\delta \ (i=1,2,\cdots,8)\}.$$

We define a holomorphic open embedding s''_t of $N(\eta) \subset U_W \subset M$ into $N(\varepsilon)$ by

$$\begin{aligned} s''_t([\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3]) = [&\mu\zeta_0+\nu\zeta_2 : (\mu t''_1+\nu t''_2)\zeta_0+(\mu+\mu t''_3+\nu t''_4)\zeta_1 \\ &+(\nu t''_1+\mu t''_2)\zeta_2+(\nu+\nu t''_3+\mu t''_4)\zeta_3 : \\ &-((\nu+\mu t''_5+\nu t''_6)\zeta_0+(\mu t''_7+\nu t''_8)\zeta_1 \\ &+(\mu+\nu t''_5+\mu t''_6)\zeta_2+(\nu t''_7+\mu t''_8)\zeta_3) : \\ &-(\nu\zeta_1+\mu\zeta_3)]. \end{aligned}$$

Assume that $\mathcal{M}(n)$ is defined with the parameter space $B^{(n)}$ and that $\mathcal{M}(n)$ contains $\overline{\iota_\lambda(U_{1/\varepsilon})} \times B^{(n)}$ such that $\overline{\iota_\lambda(U_{1/\varepsilon})} \times B^{(n)} \cap M(n) = \overline{\iota_\lambda(U_{1/\varepsilon})} \subset M(n-1)^\# \cap M^\#$. We denote $\mathcal{M}(n) - \overline{\iota_\lambda(U_{1/\varepsilon})} \times B^{(n)}$ by $\mathcal{M}(n)^\#$. We construct $\mathcal{M}(n+1)$ of $\mathcal{M}(n)^\#$ and $\mathcal{M}^\#$ by identifying

$$((x, t), t^{n+1}, t'') \in \mathcal{M}(n)^\# \times B \times B(t'')$$

with

$$((x^{n+1}, \tilde{t}^{n+1}), \tilde{t}, \tilde{t}'') \in \mathcal{M}^\# \times B^{(n)} \times B(t'')$$

if and only if

$$x = s_{t''}(x^{n+1}), \quad t=\tilde{t}, \quad t^{n+1}=\tilde{t}^{n+1}, \quad t''=\tilde{t}''.$$

We can project $\mathcal{M}(n+1)$ onto $B^{(n)} \times B \times B(t'')$ and see that $(\mathcal{M}(n+1), B^{(n)} \times B \times B(t''), \varpi)$ is a complex analytic family with $\varpi^{-1}(0) = M(n+1)$. Calculating the Kodaira-Spencer map, we get

Theorem 3. $(\mathcal{M}(n), \underbrace{B \times \cdots \times B}_{n} \times B(t') \times \underbrace{B(t'') \times \cdots \times B(t'')}_{n-2}, \varpi)$ *is the complete, effectively parametrized complex analytic family of the small deformations of* $M(n)$.

References

[1] Ma. Kato: On compact complex 3-folds with lines (preprint).

[2] K. Kodaira and D. C. Spencer: A theorem of completeness for complex analytic fibre spaces. Acta Math., **100**, 281–294 (1958).