# 104. *Invariant Polynomials on Compact Complex Manifolds*

By Akito FUTAKI*),†) and Shigeyuki MORITA††)

(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1984)

**1. Introduction.** Let $M$ be a compact complex manifold of dimension $n$, $H(M)$ the complex Lie group of all automorphisms of $M$ and $\mathfrak{h}(M)$ the complex Lie algebra of all holomorphic vector fields of $M$. In case when $c_1(M)$ is positive, the first author defined in [11] a character $f:\mathfrak{h}(M)\to \boldsymbol{C}$ which is intrinsically defined and vanishes if $M$ admits a Kähler Einstein metric.

The purpose of the present note is twofold. First we generalize the definition of $f$ to obtain a linear map $F: I^{n+k}(GL(n, \boldsymbol{C}))\to I^k(H(M))$ where, for a complex Lie group $G$, $I^p(G)$ denotes the set of all holomorphic $G$-invariant symmetric polynomials of degree $p$. The original $f$ coincides with $F(c_1^{n+1})$ up to a constant. Secondly we give an interpretation of $f$ in terms of secondary characteristic classes of Chern-Simons [9] and Cheeger-Simons [8]. More precisely we show that $f$ appears as the so-called Godbillon-Vey invariant of certain complex foliations.

We also have real analogue of the linear map $F$ for compact group actions. However this case can be derived from the recent papers by Atiyah-Bott [2] and Berline-Vergne [4], which we noticed after we have finished this work. Very interestingly both of the above works [2] [4] are inspired by Duistermaat-Heckman's paper [10] in symplectic geometry while we started from Kählerian geometry. In § 7 we shall derive a Duistermaat-Heckman type formula replacing a symplectic form and a hamiltonian vector fields by a Kähler form and a holomorphic vector field.

**2. Definition of $H(M)$-invariant polynomials.** Let $M$ be a compact complex manifold of dimension $n$. Choose any hermitian metric $h$ on $M$ and let $D$ and $\Theta$ be the Hermitian connection and the curvature form with respect to $h$ respectively. We put $L(X)=L_X-D_X$ for any $X\in\mathfrak{h}(M)$. For $\phi\in I^{n+k}(GL(n, \boldsymbol{C}))$ we define $f_\phi:(\mathfrak{h}(M))^k\to \boldsymbol{C}$ by

$$f_\phi(X_1,\cdots,X_k)=\int_M \phi\left(L(X_1),\cdots,L(X_k),\frac{i}{2\pi}\Theta,\cdots,\frac{i}{2\pi}\Theta\right).$$

---

*) Partially supported by JSPS fellowship for research abroad.

†) College of Arts and Sciences, Chiba University.

††) Department of Mathematics, College of Arts and Sciences, University of Tokyo.

**Theorem 2.1.** *The definition of $f_\phi$ does not depend on the choice of the Hermitian metric h. In particular $f_\phi$ depends only on the complex structure of M and is invariant under the coadjoint action of H(M). Therefore we have a linear map $F: I^{n+k}(GL(n, C)) \to I^k(H(M))$.*

Let $T: I^k(H(M)) \to I_A^{2k-1}(H(M))$ be the transgression operator where $I_A^p(H(M))$ denotes the set of all holomorphic $H(M)$-invariant antisymmetric polynomials of degree $p$.

**Corollary 2.2.** *For each $\phi \in I^{n+k}(GL(n, C))$ we can define $Tf_\phi \in I_A^{2k-1}(H(M))$.*

**3. Relation to the Godbillon-Vey invariant.** To begin with we prepare the following notations;

$$\det(1+tA) = 1 + tc_1(A) + \cdots + t^n c_n(A) \qquad \text{for } A \in \mathfrak{gl}(n, C),$$
$$I_0^p(GL(n, C)) = I^p(GL(n, C)) \cap Z[c_1, \cdots, c_n].$$

Consider a manifold $W = M \times S^1$ where $S^1 = R/Z$, and a vector field $Y = (\partial/\partial t) + 2\,\mathrm{Re}(X)$ on $W$ where $\mathrm{Re}(X)$ is the real part of $X \in \mathfrak{h}(M)$ and $t$ is the coordinate of $S^1$. Then the flow generated by $Y$ defines a complex foliation $\mathcal{F}$ of codimension $n$. Using Bott's vanishing theorem [6] we can define the Simons class $S_\phi(\mathcal{F}) \in H^{2n+1}(W; C/Z)$ for any $\phi \in I_0^{n+1}(GL(n, C))$, see [7] and [8]. The following motivated us to show Corollary 2.2 above and Corollary 4.2 below.

**Theorem 3.1.** $S_\phi(\mathcal{F})[W] = (n+1)(i/2\pi) f_\phi(X) \bmod Z$.

**4. Relation to the equivariant cohomology.** For brevity we write $H$ for $H(M)$. Let $EH \to BH$ be the universal $H$-bundle and $MH = EH \times_H M$. We denote by $\xi$ the vector bundle over $MH$ consisting of the vectors tangent to the fibers of the projection $\pi: MH \to BH$.

**Theorem 4.1.** *The following diagram commutes:*

$$\begin{array}{ccc} I^{n+k}(GL(n, C)) & \xrightarrow{\Phi} & I^k(H) \\ \downarrow C & & \downarrow W \\ H^{2n+2k}(MH; C) & \xrightarrow{\pi_!} & H^{2k}(BH; C) \end{array}$$

*where $\Phi = \binom{n+k}{n}\left(\frac{1}{2\pi}\right)^k F$, $\pi_!$ is the Gysin map, $C(\phi)$ is the Chern polynomial of $\xi$ corresponding to $\phi$ and $W$ is the Weil homomorphism.*

The proof is given by computing the curvature form of the associated $GL(n, C)$-bundle of $\xi$. The classes in the image of $\pi_!$ are regarded as characteristic classes of $M$-bundles. In § 6 we show some vanishing results of these classes.

Let $H^\delta$ be the group $H$ with the discrete topology, and $EH^\delta \to BH^\delta$ and $MH^\delta$ as before. Let $\phi \in I_0^{n+k}(GL(n, C))$. Theorem 4.1 asserts that $\rho\pi'_! C(\phi) = W\Phi(\phi)$ where $\pi'_!: H^{2n+2k}(MH^\delta; Z) \to H^{2k}(BH^\delta; Z)$ is the Gysin map and $\rho: H^{2k}(BH^\delta; Z) \to H^{2k}(BH^\delta; R)$ is the restriction map. Noting that $EH^\delta$ is a flat bundle, we can define $\mu: I_0^{n+k}(GL(n, C)) \to H^{2k-1}(BH^\delta; C/Z)$ by $\mu f_\phi = S_{\Phi(\phi), \pi'_! C(\phi)}$, see [8].

**Corollary 4.2.** *The following diagram commutes:*

$$\begin{array}{ccc} I_0^{n+k}(GL(n, \boldsymbol{C})) & \xrightarrow{\Phi_0} & \mathrm{Image}\,\Phi_0 \\ S\downarrow & & \downarrow\mu \\ H^{2n+2k-1}(MH^\delta ; \boldsymbol{C}/\boldsymbol{Z}) & \xrightarrow{\pi_!} & H^{2k-1}(BH^\delta ; \boldsymbol{C}/\boldsymbol{Z}) \end{array}$$

*where* $S(\phi)$ *is the Simons class defined by applying Bott's vanishing theorem to the natural complex foliation in* $MH^\delta$, *and* $\Phi_0$ *is the restriction of* $\Phi$ *to* $I_0^{n+k}(GL(n, \boldsymbol{C}))$.

**5. Localization theorem.** We say that $X \in \mathfrak{h}(M)$ is *nondegenerate* if the zero set Zero $(X)$ of $X$ consists of isolated points and at any $p \in$ Zero $(X)$ the endomorphism $L(X)_p : T_pM \to T_pM$ is nondegenerate.

**Theorem 5.1.** *Let* $X \in \mathfrak{h}(M)$ *be nondegenerate. Then for any* $\phi \in I^{n+k}(GL(n, \boldsymbol{C}))$,

$$\binom{n+k}{k} f_\phi(X) = (-1)^k \sum_{p \in \mathrm{Zero}\,(X)} \phi(L(X)_p)/\det(L(X)_p).$$

The proof of Theorem 5.1 is identical to that in [7]. This enables us to compute $f_\phi$ in concrete examples. For instance we obtain the following by investigating the blow-ups of the complex projective spaces and their products; compare with Corollary 6.3 below.

**Proposition 5.2.** *For a monomial* $\phi = c_1^{\alpha_1} \cdots c_n^{\alpha_n}$ $(\neq c_1 c_n)$ *of degree* $n+1$, $f_\phi$ *is nontrivial.*

**6. Vanishing theorems.**

**Theorem 6.1.** *If* $[\mathfrak{h}(M), \mathfrak{h}(M)] = \mathfrak{h}(M)$, *then* $f_\phi = 0$ *for any* $\phi \in I^{n+1}(GL(n, \boldsymbol{C}))$.

**Theorem 6.2.** *Let* $T_l(y ; c_1, \cdots, c_l)$ *be the generalized Todd polynomial. Let* $\phi = T_{n+k}(y ; c_1, \cdots, c_n, 0, \cdots, 0)$ *with* $k > 0$. *If M is Kähler and* $X \in \mathfrak{h}(M)$ *is nondegenerate, then* $f_\phi(X) = 0$ *for any* $y \in \boldsymbol{C}$.

**Corollary 6.3.** *Let M and X be as in Theorem* 6.2. *Then* $f_{c_1 c_n}(X) = 0$. *In particular* $\sum_{p \in \mathrm{Zero}\,(X)} \mathrm{tr}\,(L(X)_p) = 0$.

**Theorem 6.4.** *Let* $H_0$ *be the identity component of H and let* $r : H^{2k}(BH ; \boldsymbol{C}) \to H^{2k}(BH_0 ; \boldsymbol{C})$ *be the restriction map. Assume that M is Kähler and let* $\phi$ *be as in Theorem* 6.2. *Then* $r\pi_! C(\phi) = 0$ *for any* $y \in \boldsymbol{C}$.

*Sketch of proofs*: Theorem 6.1 follows from the fact that $f_\phi$ is a character for $\phi \in I^{n+1}(GL(n, \boldsymbol{C}))$. To prove Theorem 6.2 we compare our localization theorem with the Atiyah-Bott formula for the Dolbeault complex [1]. Theorem 6.4 follows from the Atiyah-Singer index theorem for families of elliptic operators [3].

**7. Duistermaat-Heckman formula.** Let $M$ be a compact complex manifold with $c_1(M) > 0$. Choose any positive (1, 1) form $\omega$ representing $c_1(M)$ and let $\gamma_\omega$ be the corresponding Ricci form, $F_\omega$ a smooth function such that $\gamma_\omega - \omega = i/2\pi\partial\bar{\partial}F_\omega$. We define $\tilde{\Delta}_\omega$ by

$$\tilde{\Delta}_\omega u = \Delta_\omega u + u^\alpha (F_\omega)_\alpha$$

where $\Delta_\omega$ is the usual complex Laplacian with respect to $\omega$. Then $\tilde{\Delta}_\omega$ is a self-adjoint elliptic operator with respect to the *weighted measure* $\exp(F_\omega)\omega^n$. We put

$$\Lambda_\omega=\{u\,;\,\tilde{\Delta}_\omega u+u=0\}.$$

**Theorem 7.1.** *$\Lambda_\omega$ is isomorphic to $\mathfrak{h}(M)$ through the correspondence $\lambda_\omega: u\mapsto\sum g^{\alpha\bar\beta}(\partial u/\partial \bar z^\beta)(\partial/\partial z^\alpha)$. Moreover if $\eta$ is the Calabi-Yau solution to $\omega=\gamma_\eta$, then we have $\Delta_\eta(\lambda_\eta^{-1}(X))=-\lambda_\omega^{-1}(X)$ for $X\in\mathfrak{h}(M)$.*

Theorem 5.1 for $\phi=c_1^{n+1}$ says that

$$\binom{n+k}{k}\int_M(\Delta_\eta(\lambda_\eta^{-1}(X)))^k\gamma_\eta^n=(-1)^{n+k}\sum_p(\Delta_\eta(\lambda_\eta^{-1}(X))_p)^{n+k}/e(p)$$

where $e(p)=\det(\partial X^\alpha/\partial z^\beta)_p$. Then by Theorem 7.1 this is equivalent to

$$\binom{n+k}{k}\int_M(\lambda_\omega^{-1}(X))^k\omega^n=(-1)^n\sum_p(\lambda_\omega^{-1}(X)_p)^{n+k}/e(p).$$

Theorem 7.2 ([10])

$$\int_M\exp(-it\lambda_\omega^{-1}(X))\frac{\omega^n}{n!}=\sum_p\frac{\exp(-it\lambda_\omega^{-1}(X)_p)}{(it)^n e(p)}.$$

## References


[1] Atiyah, M. F. and Bott, R.: A Lefschetz fixed point formula for elliptic complexes I. Ann. of Math., **86**, 374–407 (1967); II, ibid., **88**, 451–491 (1968).

[2] ——: The moment map and equivariant cohomology. Topology, **23**, 1–28 (1984).

[3] Atiyah, M. F. and Singer, I. M.: The index of elliptic operators IV. Ann. of Math., **92**, 119–138 (1970).

[4] Berline, N. and Vergne, M.: Zeros d'un champ de vecteurs et classes caracteristiques equivariantes. Duke Math. J., **50**, 539–549 (1983).

[5] Bott, R.: Vector fields and characteristic numbers. Michigan Math. J., **14**, 231–244 (1967).

[6] ——: On a topological obstruction to integrability. Proc. Symp. Pure Math., **16**, 127–131 (1970).

[7] On the Lefschetz formula and exotic characteristic classes. Symp. Math., **10**, 95–105 (1972).

[8] Cheeger, J. and Simons, J.: Differential characters and geometric invariants (preprint).

[9] Chern, S. S. and Simons, J.: Characteristic forms and geometric invariants. Ann. of Math., **99**, 48–69 (1974).

[10] Duistermaat, J. J. and Heckman, G. J.: On the variation in the cohomology of the symplectic form of the reduced phase space. Invent. Math., **69**, 259–268 (1982).

[11] Futaki, A.: An obstruction to the existence of Einstein Kähler metrics. Invent. Math., **73**, 437–443 (1983).

[12] Morita, S.: Characteristic classes of surface bundles (to appear in Bull. Amer. Math. Soc.); I (preprint), II (in preparation).

[13] Yau, S. T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. Comm. Pure Appl. Math., **31**, 339–411 (1978).