10. *Infinitesimal Deformations of Cusp Singularities*

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Introduction. The purpose of this article is to compute infinitesimal deformations T^1 of cusp singularities of two dimension. Let T be a cusp singularity, C the exceptional set of the minimal resolution of T, r the number of irreducible components of C. Then C is a (reduced) cycle of r rational curves. Our main consequence is that $\dim T^1$ is equal to $r-C^2$ if $C^2 \leqq -5$. This has been conjectured by Behnke [1]. After completing this work, I was informed that Behnke [2] solved this in a manner slightly different from ours.

§1. Definitions and a fundamental lemma. **(1.1)** Let M be a complete module in a real quadratic field K, $U^+(M)$ the group of all totally positive units keeping M invariant by multiplication, V an infinite cyclic subgroup of $U^+(M)$. We define a subgroup $G(M, V)$ of $SL(2, \boldsymbol{R})$ by

$$G(M, V)=\left\{\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} \in SL(2, \boldsymbol{R})\,;\ v \in V,\ m \in M\right\}.$$

We define an action of $G(M, V)$ on the product $\boldsymbol{H}\times\boldsymbol{H}$ of two upper half planes by

$$\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} : (z_1, z_2) \longrightarrow (vz_1+m, v'z_2+m')$$

where v' and m' denote the conjugates of v and m respectively. The action of $G(M, V)$ on $\boldsymbol{H}\times\boldsymbol{H}$ is free and properly discontinuous. We have a nonsingular surface $X'(M, V)$ as quotient. This $X'(M, V)$ is partially compactified by adding a point ∞ into a normal complex space $X(M, V)$. Let $f: Y(M, V)\to X(M, V)$ be the minimal resolution of $X(M, V)$, C the exceptional set of f, $\pi: \mathcal{D}\to Y(M, V)$ the universal covering of $Y(M, V)$, $\mathcal{C}=\pi^{-1}(C)$. For brevity we denote $X(M, V)$ and $Y(M, V)$ by X and Y respectively. The space X has a unique isolated singularity at ∞, which we call a cusp singularity. The exceptional set C is a (reduced) cycle of rational curves.

(1.2) Let M^* be the dual of M, i.e. by definition $M^*=\{x \in K\,;\ \mathrm{tr}\,(xy) \in \boldsymbol{Z}$ for any $y \in M\}$. Define a mapping i of K into $\boldsymbol{R}^2$ by $i(x)=(x, x')$, $x \in K$. Let $(M^*)^+=\{x \in M^*\,;\ x>0, x'>0\}$, and let $\Sigma^+(M)$ be the convex closure of $i((M^*)^+)$, $\partial\Sigma^+(M^*)$ be the boundary of $\Sigma^+(M^*)$. Then we number lattice points lying on $\partial\Sigma^+(M^*)$ in a consecutive order. Namely we let $i^{-1}(\Sigma^+(M^*)\cap i(M^*))=\{B_j\,;\ j \in \boldsymbol{Z}\}$ with $B_j<B_k$ for $j>k$.

The group V acts on M^*, $\Sigma^+(M^*)$ and $\partial\Sigma^+(M^*)$. Let v be a generator of V with $0<v<1$. Then there exists s such that $vB_k=B_{k+s}$ for any k. We know that $s=-C^2$ by [5]. Moreover there are positive integers b_k ($\geqq 2$) ($k\in \boldsymbol{Z}$) such that $b_{k+s}=b_k$ and $b_kB_k=B_{k-1}+B_{k+1}$ for any $k\in \boldsymbol{Z}$.

(1.3) We denote by $\Omega^1_Y(\log C)$ the sheaf over Y of germs ω of meromorphic one forms such that the poles of ω and $d\omega$ are contained in C ($=C_{\mathrm{red}}$). Since C is with normal crossing, $\Omega^1_Y(\log C)$ is locally free. In fact, $\Omega^1_Y(\log C)$ is isomorphic to $\mathcal{O}_Y(F)\oplus\mathcal{O}_Y(-F)$ for a flat line bundle F on Y. This can be shown by using natural extensions of two sections dz_1 and dz_2 to $\mathcal{D}$. Let $\tilde{\Theta}_Y(nC)=\mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_Y(\log C), \mathcal{O}_Y(nC))$. Similarly $\tilde{\Theta}_{\mathcal{D}}(nC)$ is defined.

Lemma (1.4) (Compare [1]). *Let* $B(n)=\{-aB_k-bB_{k+1}\ (\neq -bB_s);\ a>0,\ b\geqq 0,\ a+b\leqq n,\ 0\leqq k\leqq s-1\}$, $\theta(\mu)=\exp(2\pi\sqrt{-1}(\mu z_1+\mu' z_2))$. *Suppose* $s\geqq 3$.

1) *The first cohomology group* $H^1(V, H^0(\mathcal{D}, \tilde{\Theta}_{\mathcal{D}}(nC)))$ *of V-modules is generated by* $\theta(\mu)\partial_1$ *and* $\theta(\mu)\partial_2$, $\mu\in B(n)$.

2) *The first cohomology group* $H^1(V, H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(nC)))$ *of V-modules is generated by* $\theta(\mu)$, $\mu\in B(n)\cup\{0\}$.

3) *Define a homomorphism* χ: $H^1(V, H^0(\mathcal{D}, \tilde{\Theta}_{\mathcal{D}}(nC)))$ *into* $H^1(V, H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(nC)))^s$

$$\chi=(\chi_0, \chi_1, \cdots, \chi_{s-1}),$$
$$\chi_j(\theta(\mu)\partial_1)=\textstyle\sum_k' B_{j+ks}\theta(\mu+B_{j+ks}),$$
$$\chi_j(\theta(\mu)\partial_2)=\textstyle\sum_k' B'_{j+ks}\theta(\mu+B_{j+ks})$$

where Σ' *denotes the summation over the set of all* k *with* $\mu+B_{j+ks}\in -(M^*)^+\cup\{0\}$. *Then for any n large enough* $\boldsymbol{T}^1=\mathrm{Ker}\,\chi$.

Remark (1.5). In $H^1(V, H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(nC)))$, $\theta(\mu_1)=\theta(\mu_2)$ iff $V\mu_1=V\mu_2$, $\mu_k\in -(M^*)^+\cup\{0\}$.

(1.6) Let $\mu\in(M^*)^+$. Then there exist k, a and b such that $\mu=aB_k+bB_{k+1}$, $a>0$, $b\geqq 0$. These k, a and b are uniquely determined by μ. We call μ internal if $a>0$, $b>0$ and call μ k-extremal if $a>0$, $b=0$. We say that μ is extremal if μ is k-extremal for some k. We define the weight of μ by $\mathrm{wt}\,\mu=a+b$, $\mathrm{wt}\,(0)=0$. If $\mu(\neq 0)$ is not in $(M^*)^+$, then we define $\mathrm{wt}\,\mu=-\infty$. We notice that if $V\mu_1=V\mu_2$, then $\mathrm{wt}\,\mu_1=\mathrm{wt}\,\mu_2$.

Fundamental Lemma (1.7).

1) *Let* $\mu_1, \mu_2\in(M^*)^+$. *Then* $\mathrm{wt}\,(\mu_1+\mu_2)\geqq\mathrm{wt}\,(\mu_1)+\mathrm{wt}\,(\mu_2)$.

2) *Suppose that* $j_1\leqq j_2\leqq\cdots\leqq j_l$. *Then* $\mathrm{wt}\,(B_{j_1}+B_{j_2}+\cdots+B_{j_l})\geqq l+(b_{j_1+1}-2)+(b_{j_1+2}-2)+\cdots+(b_{j_l-1}-2)$. *Equality holds only when* $b_\lambda=2$ *for* $j_1+2\leqq\lambda\leqq j_l-2$.

3) *Suppose that* $j_1\leqq j_2\leqq\cdots\leqq j_l$. *Then* $\mathrm{wt}\,(B_{j_1}+B_{j_2}+\cdots+B_{j_l})=l$ *iff* $b_\lambda=2$ *for* $j_1+1\leqq\lambda\leqq j_l-1$.

Proof. Use $B_j+B_{j+n}=B_{j+1}+B_{j+n-1}+\sum_{\lambda=j+1}^{j+n-1}(b_\lambda-2)B_\lambda$ for $n\geqq 2$.

Lemma (1.8). *Suppose $s\geqq 5$, and that there is no consecutive subsequence $b_j, b_{j+1}, \cdots, b_{j+s-5}$ of b_k $(k\in Z)$ such that $b_\lambda=2$ for $j\leqq\lambda\leqq j+s-5$. Let $0\leqq i\leqq s-1$, $0\leqq j\leqq s-1$, $\mu=\alpha B_i+\beta B_{i+1}\in(M^*)^+$, $\alpha>0$, $\beta\geqq 0$, $m>1$, a, $b>0$, $c\geqq 0$, h and $k\in Z$.*

1) *If $\mu-B_{j+ks}=(m-1)B_{j+hs}$, then $\mathrm{wt}\,\mu\geqq m$, equality holding iff $h=k=0$.*

2) *If $\mu-B_{j+ks}=aB_{j-1+hs}+(b-1)B_{j+hs}$, then*

2–1) *μ is internal and $\mathrm{wt}\,\mu\geqq a+b+1$, or*

2–2) *μ is l-extremal and $\mathrm{wt}\,\mu\geqq a+b+b_l-1$, or*

2–3) *$k=h=0$, or $k=h=1$.*

3) *If $\mu-B_{j+ks}=(a-1)B_{j-1+hs}+bB_{j+hs}$, then*

3–1) *μ is internal and $\mathrm{wt}\,\mu\geqq a+b+1$, or*

3–2) *μ is l-extremal and $\mathrm{wt}\,\mu\geqq a+b+b_l-1$, or*

3–3) *$k=h=0$, or $k=h=1$.*

4) *If $\mu-B_{j+ks}=B_{j-2+hs}+cB_{j-1+hs}$, then*

4–1) *μ is internal and $\mathrm{wt}\,\mu\geqq c+3$, or*

4–2) *μ is l-extremal and $\mathrm{wt}\,\mu\geqq c+b_l+1$, or*

4–3) *$k=h=0$ and $\mu=(c+b_{j-1})B_{j-1}$ $(1\leqq j\leqq s-1)$ or $k=h=1$, $\mu=(c+b_{s-1})B_{s-1}$, $j=0$.*

5) *If $\mu-B_{j+ks}=cB_{j+1+hs}+B_{j+2+hs}$, then*

5–1) *μ is internal and $\mathrm{wt}\,\mu\geqq c+3$, or*

5–2) *μ is l-extremal and $\mathrm{wt}\,\mu\geqq c+b_l+1$, or*

5–3) *$k=h=0$ and $\mu=(c+b_{j+1})B_{j+1}$ $(0\leqq j\leqq s-2)$ or $k=h=1$, $\mu=(c+b_0)B_0$, $j=s-1$.*

§ 2. Theorem.

Theorem (2.1). *Let T be a cusp singularity with $s\geqq 5$. Then the space T^1 of infinitesimal deformations of T is, as a subspace of $H^1(V, H^0(\mathcal{D}, \tilde{\Theta}_{\mathcal{D}}(nC)))$ for n large enough, generated by*

$$\delta_{i,j}:=\theta(-iB_j)\delta_j,\quad 0\leqq j\leqq s-1,\quad 1\leqq i\leqq b_j-1$$

where $\delta_j=B'_j\partial_1-B_j\partial_2$. In particular $\dim T^1=s+r$.

Proof. For simplicity's sake we assume that there is no consecutive subsequence b_j, $b_{j+1}, \cdots, b_{j+s-5}$ of b_k such that $b_\lambda=2$ for $j\leqq\lambda\leqq j+s-5$. By (1.4) $T^1=\mathrm{Ker}\,\chi$. Take $\xi\in\mathrm{Ker}\,\chi$. Express

$$\xi=\textstyle\sum_{\mu\in B}\theta(\mu)(C(\mu)\partial_1+D(\mu)\partial_2)$$

for a finite subset B of $B(n)$ and constants $C(\mu)$ and $D(\mu)$. Define

$$h(B)=\max\left\{\mathrm{wt}\,(-\mu)-b_i\,;\ \begin{array}{l}\mu(\in B)\text{ is }i\text{-extremal for some }i\\ \text{either }C(\mu)\neq 0\text{ or }D(\mu)\neq 0\end{array}\right\}.$$

First we prove

Lemma (2.2). *Suppose $h(B)\geqq 0$. Then $C(\mu)=D(\mu)=0$ if μ is internal and if $\mathrm{wt}\,(-\mu)\geqq h(B)+2$.*

Proof of Lemma (2.2). Let $l=\max\{\mathrm{wt}\,(-\mu)\,;\,\mu(\in B)$ is internal, either $C(\mu)\neq 0$ or $D(\mu)\neq 0\}$. Then we may assume $l\geqq h(B)+2$. Then by (1.8)

$$\begin{aligned}\chi_j(\xi)=&\sum_{a,b>0}^{a+b=l}\theta(-aB_{j-1}-(b-1)B_j)(C(-aB_{j-1}-bB_j)B_j\\&\qquad+D(-aB_{j-1}-bB_j)B'_j)\\&+\sum_{a,b>0}^{a+b=l}\theta(-(a-1)B_j-bB_{j+1})(C(-aB_j-bB_{j+1})B_j\\&\qquad+D(-aB_j-bB_{j+1})B'_j)\\&+(\text{terms for }\mu\neq aB_{j-1}-(b-1)B_j,\ -(a-1)B_j-bB_{j+1},\\&\qquad a+b=l,\ a,\ b>0).\end{aligned}$$

Hence we have

$$C(-aB_{j-1}-bB_j)B_j+D(-aB_{j-1}-bB_j)B'_j=0,$$
$$C(-aB_{j-1}-bB_j)B_{j-1}+D(-aB_{j-1}-bB_j)B'_{j-1}=0.$$

Since $B_jB'_{j-1}-B'_jB_{j-1}\neq 0$, we have $C(-aB_{j-1}-bB_j)=D(-aB_{j-1}-bB_j)=0$ for $a+b=l$, a, $b,>0$. This contradicts the definition of l, hence (2.2) is proved. Q.E.D.

Let $m_j=h(B)+b_j$. By the definition of $h(B)$, $C(-mB_j)=D(-mB_j)=0$ if $m\geqq m_j+1$. If $h(B)\geqq 0$, then by (2.2) and (1.8)

$$\begin{aligned}\chi_j(\xi)=&\theta(-(m_j-1)B_j)(C(-m_jB_j)B_j+D(-m_jB_j)B'_j)\\&+\theta(-h(B)B_{j-1}-B_{j-2})(C(-m_{j-1}B_{j-1})B_j+D(-m_{j-1}B_{j-1})B'_j)\\&+(\text{terms for }\mu\neq-(m_j-1)B_j,\ -h(B)B_{j-1}-B_{j-2}).\end{aligned}$$

Hence

$$C(-m_jB_j)B_j+D(-m_jB_j)B'_j=C(-m_jB_j)B_{j+1}+D(-m_jB_j)B'_{j+1}=0$$

from which it follows $C(-m_jB_j)=D(-m_jB_j)=0$. This contradicts the definition of $h(B)$. Hence $h(B)$ is negative. Then by the same argument as above $C(\mu)=D(\mu)=0$ for μ internal, so that

$$\xi=\sum_{j=0}^{s-1}\sum_{i=1}^{b_j-1}\theta(-iB_j)(C-iB_j)\partial_1+D(-iB_j)\partial_2).$$

Then

$$\chi_j(\xi)=\sum_{i=0}^{b_j-1}\theta(-(i-1)B_j)(C-(-iB_j)B_j+D(-iB_j)B'_j)$$

which shows (2.1). Theorem in the general case can be proved similarly by using (1.7). Thus $\dim \boldsymbol{T}^1=s+\sum_{\lambda=0}^{s-1}(b_\lambda-2)=s+r$ by [5], where $r=\sharp$ (irreducible components of C). Q.E.D.

The same method yields a complete description of $\boldsymbol{T}^1$ as a subspace of $H^1(V, H^0(\mathcal{D}, \tilde{\Theta}_{\mathcal{D}}(nC)))$ in the cases $1\leqq s\leqq 4$. The details will appear elsewhere [4].

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