96. On Differential Operators and Congruences for Siegel Modular Forms of Degree Two

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§ 1. **Introduction.** We study congruences between Siegel modular forms of degree two and different weight by using differential operators. In the degree one case, such congruences were studied by Serre [6] and Swinnerton-Dyer [8]. For the degree two case, we refer to Kurokawa [2]. We denote by $M_k(\Gamma_n)$ (resp. $M_k^\infty(\Gamma_n)$, $S_k(\Gamma_n)$) the $\boldsymbol{C}$-vector space of holomorphic Siegel modular forms (resp. C^∞-modular forms, holomorphic cusp forms) of degree n and weight k. For a subring R of $\boldsymbol{C}$, we denote by $M_k(\Gamma_n)_R$ the R-submodule of $M_k(\Gamma_n)$ consisting of Siegel modular forms which have Fourier coefficients in R. This paper is an abstract of [5].

§ 2. **General results.** We introduce certain differential operators. For a variable $Z=\begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ on H_2 of Siegel upper half plane of degree two, we put

$$Y=\frac{1}{2i}(Z-\bar{Z})=\begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}, \qquad \frac{d}{dZ}=\begin{pmatrix} \dfrac{\partial}{\partial z_1} & \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} \\ \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} & \dfrac{\partial}{\partial z_2} \end{pmatrix}$$

and $dY=dy_1dy_2dy_3$. For integers k and $r\geqq 0$, we define a differential operator δ_k acting on a C^∞-function f on H_2 by

$$\delta_k f=|Y|^{-k+(1/2)}\left|\frac{d}{dZ}\right|(|Y|^{k-(1/2)}f)$$

and put $\delta_k^r=\delta_{k+2r-2}\cdots\delta_{k+2}\delta_k$. We understand that δ_k^0 is the identity operator. These differential operators were studied by Maass [4]. By Harris [1, 1.5.3], δ_k^r maps $M_k^\infty(\Gamma_2)$ to $M_{k+2r}^\infty(\Gamma_2)$.

Next, we make a survey of a holomorphic projection. We set $V=\{Y\in M(2,\boldsymbol{R})\,|\,Y>0\}$. For $f\in M_w^\infty(\Gamma_2)$, let $f(Z)=\sum_T a(T,Y,f)q^T$ be its Fourier expansion, where $q^T=\exp(2\pi i\,\mathrm{Tr}\,(TZ))$ and T runs over all half-integral matrices of size two. We put

$$P_w(f)=\sum_{T>0}P(w,T,a(T,Y,f))q^T,$$

where

$$P(w,T,a(T,Y,f))=\frac{\int_V a(T,Y,f)e^{-4\pi\,\mathrm{Tr}\,(TY)}|Y|^{w-3}dY}{\int_V e^{-4\pi\,\mathrm{Tr}\,(TY)}|Y|^{w-3}dY}$$

and T runs over all half-integral positive definite matrices. Then, $P_w(f)$ belongs to the ring of formal power series $C[q_3, q_3^{-1}][[q_1, q_2]]$ where $q_j=\exp(2\pi i z_j)$. It is known that if f is of bounded growth in the sense of Sturm [7, § 2(6)], then $P_w(f)$ converges for all $Z \in H_2$ and it is a holomorphic cusp form of weight w. (See Sturm [7, Theorem 1].) For complex numbers α and β, we put

$$\varepsilon(\alpha, \beta)=\begin{cases}\alpha(\alpha-1)\cdots(\beta+1)\beta & \text{if } \alpha-\beta \text{ is a non-negative integer,}\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\eta(\alpha, \beta)=\begin{cases}\alpha\left(\alpha-\dfrac{1}{2}\right)\cdots\left(\beta+\dfrac{1}{2}\right)\beta & \text{if } 2(\alpha-\beta) \text{ is a non-negative integer,}\\ 1 & \text{otherwise.}\end{cases}$$

Theorem 1. *Let R be a subring (not necessarily containing 1) of C satisfying $(1/2)R\subset R$. Let $f \in M_{k_1}(\Gamma_2)_R$ and $g \in M_{k_2}(\Gamma_2)_R$ with $k_1+k_2>4$. Suppose that I is an ideal of R satisfying*

(1) *$(1/2)I\subset I$,*

(2) *$a(T, g) \in I$ for all $T\neq 0$.*

Let r_1 be a non-negative integer and r_2 be a positive integer. We put $r=r_1+r_2$ and $w=k_1+k_2+2r$. Then for any positive integer m,

$$(2\pi i)^{-2r}\xi a(mE, P_w(\delta_{k_1}^{r_1}f\cdot\delta_{k_2}^{r_2}g))-\nu m^{2r}a(mE, fg)$$

belongs to $(2w-2r-3)I$, where $\xi=\varepsilon(w-3, w-r-2)\varepsilon(w-(5/2), w-r-(3/2))$ and $\nu=\eta(k_1+r_1-1, k_1-(1/2))\eta(k_2+r_2-1, k_2-(1/2))$.

Theorem 2. *Let $f \in M_k(\Gamma_2)$ and $g \in M_l(\Gamma_2)$ with $w>4$ where $k+l=w$. Let r and s be non-negative integers. Then we have the following:*

(1) *$\delta_k^r f\cdot\delta_l^s g$ is of bounded growth for $r+s\geqq 3$. Especially, $P_{w+2r}(g\delta_k^r f)$ belongs to $S_{w+2r}(\Gamma_2)$ for $r\geqq 3$.*

(2) *If at least one of f and g is a cusp form, then $\delta_k^r f\cdot\delta_l^s g$ is of bounded growth for all $r, s\geqq 0$.*

(3) *$P_{w+2}(g\partial_k f+f\partial_l g)$ belongs to $S_{w+2}(\Gamma_2)$ where $\partial_k^r=\varepsilon(k+r-(3/2), k-(1/2))^{-1}\delta_k^r$. Especially, $P_{2k+2}(f\delta_k f)$ belongs to $S_{2k+2}(\Gamma_2)$.*

(4) *$P_{w+4}(g\partial_k^2 f+2\partial_k f\cdot\partial_l g+f\partial_l^2 g)$ belongs to $S_{w+4}(\Gamma_2)$.*

For each integer $m\geqq 1$, $T(m): M_k(\Gamma_n)\to M_k(\Gamma_n)$ denotes the m-th Hecke operator. If $n\leqq 2$ and f is a non-zero eigen function of all Hecke operators $T(m)$, we call f an eigen form and denote the eigenvalue of $T(m)$ by $\lambda(m, f)$.

Theorem 3. *Let K be an algebraic number field, O_K be its ring of integers, $\mathfrak{p}$ be its prime ideal not dividing the ideal (2), and R be the localization of O_K at $\mathfrak{p}$. Let $f \in M_{w-2r}(\Gamma_2)_R$ and $g \in S_w(\Gamma_2)_R$ be eigen forms with $4<w-2r<w$. Suppose that all the following conditions (1)–(6) are satisfied:*

(1) *There exist positive integers $m_1, \cdots, m_n$ such that*

$$N_{L/K}|(\lambda(m_i, f_j))_{1\leqq i,j\leqq n}|\not\equiv 0 \mod \mathfrak{p}$$

where $n=\dim S_w(\Gamma_2)$ *and* $\{f_1, \cdots, f_n\}$ *is an eigen basis of* $S_w(\Gamma_2)$ *and* L *is the composite field of* K *and* $\boldsymbol{Q}(\lambda(m, f_j)\,|\,m\geqq 1)$ *for* $j=1, \cdots, n$.

(2) *There exist a positive integer* e *and* $2s$ ($s\geqq 1$) *modular forms* $h_{1,t}\in M_{k_{1,t}}(\Gamma_2)_R$, $h_{2,t}\in M_{k_{2,t}}(\Gamma_2)_R$ *with* $k_{1,t}+k_{2,t}=w-2r$, $r_{1,t}\geqq 0$, $r_{2,t}\geqq 1$ *and* $r_{1,t}+r_{2,t}=r$ *for* $t=1, \cdots, s$ *such that*

$$a(mE, f)\equiv a\Big(mE, \sum_{t=1}^{s}\nu_t h_{1,t}h_{2,t}\Big) \mod \mathfrak{p}^e$$

for all $m\geqq 1$, *where*

$$\nu_t=\eta\Big(k_{1,t}+r_{1,t}-1,\ k_{1,t}-\frac{1}{2}\Big)\eta\Big(k_{2,t}+r_{2,t}-1,\ k_{2,t}-\frac{1}{2}\Big).$$

(3) $\mathfrak{p}^e$ *divides* $(2w-2r-3)I$ *where* I *is the ideal of* R *generated by* $a(T, h_{2,t})$ *for* $T\geqq 0$, $T\neq 0$ *and* $t=1, \cdots, s$.

(4) $a(E, f)\equiv a(E, g) \mod \mathfrak{p}^e$ *and* $a(E, f)\not\equiv 0 \mod \mathfrak{p}$

(5) $m_i^{2r}\lambda(m_i, f)\equiv \lambda(m_i, g) \mod \mathfrak{p}^e$ *for* $i=1, \cdots, n$.

(6) $\sum_{t=1}^{s} P_w(\delta_{k_{1,t}}^{r_{1,t}}h_{1,t}\cdot\delta_{k_{2,t}}^{r_{2,t}}h_{2,t})$ *belongs to* $S_w(\Gamma_2)$.

Then we have:

$$m^{2r}\lambda(m, f)\equiv\lambda(m, g) \mod \mathfrak{p}^e \quad \textit{for all } m\geqq 1.$$

§3. Examples. We prove some congruences between Siegel modular forms of degree two and different weight by using Theorem 3. Let Φ be the Siegel Φ-operator. For an eigen form $f\in M_k(\Gamma_1)$, there is a unique eigen form $[f]\in M_k(\Gamma_2)$ such that $\Phi[f]=f$. Let σ_k be Saito-Kurokawa lifting $M_{2k-2}(\Gamma_1)\to M_k(\Gamma_2)$. Let $S_k^{II}(\Gamma_2)$ be the orthogonal complement of $\sigma_k(S_{2k-2}(\Gamma_1))$ in $S_k(\Gamma_2)$ with respect to the Petersson inner product. We may call an element of $S_k^{II}(\Gamma_2)$ a generic form since it does not lie in the image of Eisenstein lifting and Saito-Kurokawa lifting. We use the usual notation for modular forms such as Δ_k, χ_k and φ_k. The modular form $\chi_{20}^{(3)}\in S_{20}^{II}(\Gamma_2)$ defined by $4\chi_{10}\varphi_4\varphi_6-12\chi_{12}\varphi_4^2+28569600\chi_{10}^2$ has the minimal weight 20 among generic forms. (See Kurokawa [3, § 5].) By using Theorem 3, we have the following congruences.

Theorem 4. *The following congruences hold for all* $m\geqq 1$:

$$\begin{aligned} \lambda(m, \chi_{20}^{(3)})&\equiv m^2\lambda(m, [\Delta_{18}]) && \mod 7,\\ \lambda(m, \chi_{10})&\equiv m^2\lambda(m, \varphi_8) && \mod 5, \qquad (*)\\ \lambda(m, \chi_{12})&\equiv m^4\lambda(m, \varphi_8) && \mod 17,\\ \lambda(m, \chi_{14})&\equiv m^6\lambda(m, \varphi_8) && \mod 19. \end{aligned}$$

Remark. In the proof of (*), we use Theorem 3 with slight modification.

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